CURRENT COMMUTATORS AND SUM RULES IN ELEMENTARY PARTICLE THEORIES

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by

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C O N T E N T S

CONTENTS(Contd.)

PREFACE

The work presented in the following pages deals with certain aspects of the current algebraic approach to the problem of broken symmetries in the theory of elementary particles (Chapter 1) and related considerations arising from the attempts to understand the situation in field theoretic terms (Chapter 2). In the final chapter we are concerned mainly with an investigation into the possible superconvergence relations for pion photoproduction off nucleons through a study of the known Regge trajectories arising in the u-channel. Such relations have recently attracted considerable interest because of their relevance to the elucidation of higher symmetry schemes and the algebra of currents.

The work contained in Sections 7-15, Chapter 1; Sections 5-6 Chapter 2; Sections 4-5, Chapter 3 is original.

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INTRODUCTION

The thesis comprises of three chapters, the contents of which are interrelated a good deal. The first chapter begins with an introduction to the notion of currents in the context of weak interaction theory. Under certain assumptions the algebra of hadron 'charges' is derived. It is explained how the usual ideas of the local operator theory do not allow us to speak meaningfully of the symmetry group (rather than the symmetry algebra) for broken symmetries. Certain limitations of these arguments are indicated. It is pointed out that the situation of approximate mass degeneracy and approximate universality of meson baryon couplings is nevertheless described by exact algebraic relations between charges and currents. From the equal time charge current commutation relations and the PCAC hypothesis, certain relations between the various form factors for the hyperons and their resonances are derived. Some interesting consequences are deduced from these relations. It is shown that we must require $F_\pi^0(t) = 1$, where $F_\pi^0(t)$ is the pion electromagnetic form factor extrapolated to zero pion mass, if we are to apply consistently the PCAC and the equal time charge current commutation relations. The ratio of the symmetric (D) type to antisymmetric (F) type of meson baryon couplings is calculated in the decuplet approximation. The value thus obtained $(D/F = 3)$ is in good agreement with other estimates of the ratio, referred to in the text. This serves to strengthen our confidence in the decuplet approximation which is then used for estimating the magnetic moments of the hyperons.

The values for the magnetic moments of Σ^+ , Λ thus obtained, are in good agreement with experiment. It is also shown that within the context of the same approximation scheme, consistency can be achieved only if we assume that $\Lambda_\beta(1405)$ belongs to an SU(3) singlet.

In the second chapter the definition of currents as bilinears in the fermi field is examined. The necessity for modifying the definition of the currents as bilinears in the field at the same point is demonstrated. To account for the non-commutativity of the electromagnetic current with the transverse electric field, the usual form for the current as the limit of a gauge invariant non local operator (containing explicit field dependence) is given. It is then shown that the bilinears in the fermi field occurring in the σ -model for PCAC, must be modified in an analogous manner in order that a certain physically undesirable feature of the model (viz., the existence of a singularity at $s=0$ in one of the spectral functions), may be excluded.

In the last chapter it is shown that the assumptions of equal time current current commutation relations, covariance of the retarded products of currents together with suitable pole dominance hypothesis concerning the matrix elements of currents, enable us to write certain simple relations for amplitudes describing strong interactions. These relations (called super-convergence relations), are alternatively deducible from assumptions of analyticity of the amplitudes and appropriate high energy behaviour following from the usual ideas of Regge pole theory. Such relations have been obtained for pion photo-production off nucleons.

CHAPTER I

EQUAL TIME COMMUTATION RELATIONS FOR CURRENTS AND CHARGES, AND SOME OF THEIR CONSEQUENCES.

1. Introduction

The importance of currents (bilinears) constructed from fields) was first recognised in the context of weak interaction theory where it became apparent that a current-current form of the interaction Lagrangian density correctly describes weak interactions involving leptons. The form of the Lagrangian density $\mathcal{L}_W$ is usually taken to be

$$\mathcal{L}_W = G[l^\mu l_\mu^{\dagger} + J^\mu l_\mu + J^{\mu\dagger} l_\mu^{\dagger}] + \mathcal{L}_{n.l.} \tag{1.1}$$

where $\mathcal{L}_{n.l.}$ describes non leptonic weak interactions, l_μ is the lepton current and J_μ is the hadron current. The explicit form of l_μ in terms of the lepton fields is

$$l_\mu = i\,\bar{\mu}\gamma_\mu(1+\gamma_5)\nu_\mu + i\,\bar{e}\gamma_\mu(1+\gamma_5)\nu_e \tag{1.2}$$

Cabibbo(1) has proposed that the hadron current J_μ is a linear combination of currents which transform as a regular representation of the algebra SU3 (generated by space integrals of time components of the currents), viz.

$$J = \cos\theta\,(J^1 + iJ^2) + \sin\theta\,(J^4 + iJ^5)$$
$$J^i = V^i + A^i \tag{1.3}$$

V^i, A^i transform as a vector and axial vector respectively. This form of the hadron current will be derived in the following from a set of simpler assumptions. It can be seen that J_μ consists of two parts, a hypercharge conserving current with $|\Delta I| = 1$ and a current with $\Delta Y = \Delta Q_H = \pm 1$, $|\Delta I| = \frac{1}{2}$. The existence of these two parts of the current is well established experimentally. The evidence for or against the existence of other possible parts is not very conclusive at present. The existing experimental evidence is in accord with the local four point interaction given above, although it is possible that the coupling is not quite local but mediated by vector bosons of high mass in which case the structure of weak interactions would bear formal resemblance to that of strong and electromagnetic interactions. The intermediate boson picture suggests that $\mathcal{L}_{n.l.}$ contains at least a term of the form

$$G\, J_\mu^\dagger J^\mu$$

which would give rise to non leptonic interactions with $\Delta Y = 0$ and $\Delta Y = \pm 1$. The latter $(\Delta Y = \pm 1)$ are well established experimentally whereas the former interactions $(\Delta Y = 0)$ have been searched for by testing parity mixing in nuclear levels and parity violating effects in reactions of the type $n + \text{nucleus} \rightarrow \text{nucleus} + \gamma$ using polarised neutrons. The upper limits thus obtained on the strength of such interactions are at present approximately at the level of the expected strength of the interactions. The presence of $\Delta Y = -\Delta Q_H$ terms in the hadronic current would give rise to $\Delta Y = 2$ non leptonic transitions causing a mass difference $m(K_s) - m(K_l) \sim 1\,\text{e.v.}$ This would be incompatible with the well established experimental value of the mass difference[2] $m(K_l) - m(K_s) \sim 4 \times 10^{-6}\,\text{e.v.}$ indicating a second order weak interaction. The form given above for $\mathcal{L}_{n.l.}$ has the disadvantage that it gives rise to $\Delta Y = 1$ transitions with both $|\Delta I| = \frac{1}{2}$ and $|\Delta I| = \frac{3}{2}$. In contrast, a striking feature of non leptonic transitions is the almost complete absence of $|\Delta I| = \frac{3}{2}$ amplitudes.

This anomaly can be explained in two ways: a) the $|\Delta I| = 3/2$ part is suppressed for some dynamical reason, as has been shown that the algebra of currents approach suggests such a mechanism of suppression for some of the relevant cases, b) $\mathcal{L}_{n.l.}$ contains a part arising from the self coupling of an appropriate neutral hadronic current N_λ so that the total non leptonic lagrangian gives explicitly the $|\Delta I| = 1/2$ selection rule.

2. Some Aspects of the Phenomenology of Weak Currents.

The best known weak transitions, $n \to p + e^- + \bar{\nu}_e$, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ are described by the effective Lagrangian

$$G\, l_\mu^\dagger l^\mu + G_V \left(V^{\mu +} l_\mu^\dagger + V^{\mu -} l_\mu \right) - G_A \left(A^{\mu +} l_\mu^\dagger + A^{\mu -} l_\mu \right) \tag{2.1}$$
$$V_\mu^\pm = V_\mu^1 \pm i V_\mu^2$$

where $\frac{G_V}{G} \simeq .978$, $-\frac{G_A}{G} \simeq 1.16 \pm .04$ from experimental data. The approximate equality of G_V and G can be understood in the framework of two assumptions a) universality of the fundamental weak interaction, b) conservation of the vector part of the hadronic current (CVC). The latter hypothesis is due to Gerstein and Zeldovitch[3] (and to Feynman and Gell Mann[4]). Apart from the result that the vector coupling constant is not renormalised by strong interactions, the (CVC) hypothesis has many other interesting consequences e.g. (α) the rate of decay $\pi^+ \to \pi^0 + e^+ + \nu_e$ is completely determined by G_V . The T matrix element for the process is

$$T = i G_V\, \bar{u}_e \gamma_\mu (1+\gamma_5) u_\nu \langle \pi^0 | V^\mu | \pi^+ \rangle, \quad \langle \pi^0 | V^\mu | \pi^+ \rangle = -\sqrt{2} F_\pi(q^2) (p_{\pi^0} + p_{\pi^+}) \tag{2.2}$$

where $q = p_{\pi^+} - p_{\pi^0}$, $F_\pi(0) = 1$

$$\langle \pi^0 | A^\mu | \pi^+ \rangle = 0 \tag{2.3}$$

The unique form factor $F_\pi(q^2)$ is a consequence of the conserved vector current hypothesis. The ratio of the theoretical value of $\Gamma(\pi^+ \to \pi^0 + e^+ + \nu_e)$ to the experimental value of $\Gamma(\pi^+ \to \mu^+ + \nu_\mu)$ is

$$\frac{\Gamma(\pi^+ \to \pi^0 + e^+ + \nu_e)_{th.}}{\Gamma(\pi^+ \to \mu^+ + \nu_\mu)_{exp.}} \sim 1.00 \times 10^{-8}$$

This compares favourably with the ratio of experimental values,

$$\frac{\Gamma(\pi^+ \to \pi^0 + e^+ + \nu_e)_{exp.}}{\Gamma(\pi^+ \to \mu^+ + \nu_\mu)_{exp.}} \sim 1.13 \pm .09 \times 10^{-8}$$

The uncertainty in the above ratio is entirely due to the uncertainty in $\Gamma(\pi^+ \to \pi^0 + e^+ + \nu_e)_{exp.}$.

(β) The matrix element of the conserved vector current for the nucleons is specified by two form factors, i.e.

$$\langle p'\lambda' | V_\mu^\pm | p\lambda \rangle = \bar{u}_N(p',\lambda')\left[i\gamma_\mu F_1^V(q^2) - \frac{1}{2m}\sigma_{\mu\nu} q^\nu F_2^V(q^2)\right]\frac{\tau^\pm}{2} u_N(p,\lambda)$$

$$\tau^\pm = \tau^1 \pm i\tau^2, \quad q = p' - p, \quad F_1^V(0) = 1, \quad F_2^V(0) = \mu_p - \mu_n \tag{2.4}$$

μ_p and μ_n are the anomalous proton and neutron magnetic moments respectively. The 'weak magnetism' term has been established through its effect on the electron energy spectrum in β-decay. The isovector part of the electromagnetic current is assumed to be equal to the neutral component of the isovector current whose charged components are the $\Delta Y = 0$ weak vector currents. It will be shown in the following that the above assumption along with two other assumptions concerning the vector currents lead us to conclude that the isoscalar part of the electromagnetic current and the $\Delta Y = \pm 1$ weak vector currents can be adjoined to the above mentioned three currents giving a regular representation of the SU(3) algebra which is the algebra of hadron vector currents. In particular,

$$V_\mu^{elect.} = V_\mu^3 + \frac{1}{\sqrt{3}} V_\mu^8 \tag{2.5}$$

We have been speaking of the algebra of hadron vector currents rather than the group for reasons to be elucidated in the following.

For the sake of completeness we shall briefly mention the fact that the vector currents $\Delta Y = \Delta Q_H = \pm 1$ are not conserved. This is because the decay of $K^+ \to \pi^0 + \mu^+ + \nu_\mu$ which arises due to the coupling of such a current to the leptonic current is described by two form factors, $f^+_{\pi K}$, $f^-_{\pi K}$ such that $f^+_{\pi K}(0) \neq 0$

$$T = \ell^\lambda \langle \pi^0 | V_\lambda | K^+ \rangle$$
$$\langle \pi^0 | V_\lambda | K^+ \rangle = f^+_{\pi K}(q^2)(p_\pi + p_K) + f^-_{\pi K}(q^2)(p_\pi - p_K), \quad q = p_\pi - p_K \tag{2.6}$$

The decay of the pion similarly indicates lack of conservation of the axial vector current[5].

3. <u>'Derivation' of the Algebra of Hadron Charges</u>.

In this 'derivation' first given by Radicati[6] the following experimental facts are used:

a) The strong interactions are invariant under the transformations associated with the isospin algebra $(SU2)_I$. There exists an additive quantum number, the hypercharge Y which is conserved in strong interactions. It commutes with the generators I_i of $(SU2)_I$ and with Q_H, Q_L, the hadron and the lepton charges respectively.

b) Q_H, Y, I_3 are related by the Gell-Mann Nishijima equation:

$$Q_H = I_3 + \frac{Y}{2} \tag{3.1}$$

c) Semileptonic processes are governed by the selection rules:

$$\Delta Y = 0 \ , \ \Delta Q_H = -\Delta Q_L = 0$$
$$\Delta Y = \Delta Q_H = -\Delta Q_L = \pm 1 \tag{3.2}$$

The $\Delta Y = 0$ vector transitions are produced by a conserved current.

The explicit form for ℓ_λ and Q_L in terms of the lepton fields e, μ, ν_e, ν_μ is well known. Thus $Q_L = \int \ell_0^{el.m.}(x)\, d\underline{x}$ where

$$\ell_\lambda^{el.m.} = -i\left(\bar{e}\gamma_\lambda e + \bar{\mu}\gamma_\lambda \mu\right)$$

$$\ell_\lambda^{\pm} = \frac{i}{2}\left[\bar{e}\gamma_\lambda(1\pm\gamma_5)e + \bar{\mu}\gamma_\lambda(1\pm\gamma_5)\mu\right] \tag{3.3}$$

The canonical equal time anticommutation relations for the lepton fields, which follow from Schwinger's Action Principle[7] and the requirement of symmetry under time inversion, enable us to write

$$\left[Q_L, \ell_\lambda^{\pm}(x)\right] = \pm\, \ell_\lambda^{\pm}(x) \tag{3.4}$$

The explicit form for the hadron current J_λ and Q_H in terms of the physical or fundamental hadron fields is not known at present. However, we may write in analogy with the leptonic case

$$\left[Q_H, J_\lambda^{\pm}(x)\right] = \pm\, J_\lambda^{\pm}(x) \tag{3.5}$$

This relation merely expresses the fact that $J_\lambda^{\pm}$ is a step operator for Q_H. The hadron current can be written in terms of the vector and axial vector parts as

$$J_\lambda^{\pm} = V_\lambda^{\pm} + A_\lambda^{\pm} \tag{3.6}$$

Define $$G^{\pm} = \int V_0^{\pm}\, d\underline{x} \tag{3.7}$$

Since $J_\lambda^{\pm}$ causes transitions with $\Delta Y = 0$ and $\Delta Y = \Delta Q_H = \pm 1$ we can decompose $G^{\pm}$ as follows

$$G^+ = \alpha I^+ + \beta K^+ \tag{3.8a}$$

$$G^- = \alpha^* I^- + \beta^* K^- \tag{3.8b}$$

where

$$[Q_H, I^{\pm}] = \pm I^{\pm} \tag{3.9}$$

$$[Q_H, K^{\pm}] = \pm K^{\pm} \tag{3.10}$$

$$[Y, I^{\pm}] = 0 \tag{3.11}$$

$$[Y, K^{\pm}] = \pm K^{\pm} \tag{3.12}$$

The operators $I^{\pm}$ introduced above are constants of motion (in the absence of electromagnetic interactions) because the $\Delta Y = 0$ vector current is conserved. The operators obviously commute with Y, parity operator $\mathcal{P}$, generators of the inhomogeneous Lorentz group and act as step operators for Q_H. The only constants of motion with the above commutation properties are indeed the generators $I^{\pm}$ of the invariance algebra $(SU2)_I$ of strong interactions, if we assume that our present knowledge of quantum numbers for the specification of a state is complete.

The algebra of 'charge' operators $I^{\pm}$, $I_3 = \frac{1}{2}[I^+, I^-]$, $K^{\pm}$ will now be derived under the following assumptions.

1) The algebra is a Lie algebra, containing no operators violating the selection rules given above.

2) The operators G^+, G^-, $G_3 = \frac{1}{2}[G^+, G^-]$ form an (SU2) algebra.(6)

From Eqs. (1,10,12) we have

$$[I_3, K^{\pm}] = \pm \tfrac{1}{2} K^{\pm} \tag{3.13}$$

The assumption (1) requires

$$[I^{\pm}, K^{\pm}] = 0 \quad (3.14) \qquad [I^{\pm}, [I^{\pm}, K^{\mp}]] = 0 \tag{3.15}$$

Define $\quad L^{\pm} = \pm [I^{\mp}, K^{\pm}]$ (3.16)

$L^{\pm}$ satisfy the following commutation relations.

$$[Q_H, L^{\pm}] = 0 \ (3.17) \quad [I_3, L^{\pm}] = \mp\tfrac{1}{2} L^{\pm} \ (3.18) \quad [I^{\pm}, L^{\pm}] = \pm K^{\pm} \ (3.19)$$

$$[Y, L^{\pm}] = \pm L^{\pm} \quad (3.20)$$

The equations (13,18,19) indicate that (K^+, L^+) transform as a spinor and $(L^-, -K^-)$ transform as the conjugate spinor under operations of the algebra $(SU2)_I$. The above equations also state that the operators $K^{\pm}, L^{\pm}, I^{\pm}$ close the algebra. We have thus 'deduced' the so called $|\Delta I| = \frac{1}{2}$ rule for semileptonic processes as a consequence of assumptions ($1,2$).

Define $\quad K_3 = \frac{1}{2}[K^+, K^-]$ (3.21)

Since K_3 commutes with Q_H and Y we can write

$$K_3 = a Q_H + b Y \tag{3.22}$$

so that

$$[K_3, K^+] = (a+b) K^+ \tag{3.23}$$

$$[K_3, I^+] = a I^+ \tag{3.24}$$

$$[L^-, K^+] = -2a I^+ \tag{3.25}$$

Now,

$$G_3 = \tfrac{1}{2}[G^+, G^-]$$
$$= |\alpha|^2 I_3 + |\beta|^2 K_3 + \tfrac{1}{2}\beta\alpha^*[K^+, I^-] + \tfrac{1}{2}\alpha\beta^*[I^+, K^-] \tag{3.26}$$

$$[G_3, G^+] = \alpha\{|\alpha|^2 + 2|\beta|^2 a\} I^+ + \beta\{|\alpha|^2 + |\beta|^2(a+b)\} K^+ \tag{3.27}$$

From assumption (2) we therefore have

$$|\alpha|^2 + 2|\beta|^2 a = |\alpha|^2 + |\beta|^2(a+b) = 1 \tag{3.28}$$

so that $a = b$, $|\alpha|^2 + 2|\beta|^2 a = 1$

Redefining $K^{\pm}$ as $\frac{K^{\pm}}{\sqrt{2a}}$ and K_3 as $\frac{K_3}{2a}$ we finally obtain

$$[K^+, K^-] = 2K_3 = Q_H + Y \tag{3.21'}$$

$$[K_3, K^+] = K^+ \tag{3.23'}$$

$$[K_3, I^+] = \frac{1}{2} I^+ . \tag{3.24'}$$

With this redefinition of $K's$ we also get from Eqns. (27,28) $|\alpha|^2 + |\beta|^2 = 1$ which is the Cabibbo universality condition.

We can thus write

$$G^{\pm} = \cos\theta\, I^{\pm} + \sin\theta\, K^{\pm} \tag{3.29}$$

$$G_3 = \cos^2\theta\, I_3 + \sin^2\theta\, K_3 - \frac{1}{2}\sin\theta\cos\theta\,(L^+ + L^-) \tag{3.29a}$$

Finally, define $L_3 = \frac{1}{2}[L^+, L^-]$ (3.30)

$[L_3, L^+]$ commutes with Q_H and is a step operator for Y .

$$\therefore \quad [L_3, L^+] = cL^+ \tag{3.31}$$

$[L^+, K^-] = I^-$ (3.32) from Eqn. (25) and the redefinition of $K^{\pm}$.

Alternatively we can write

$$[L^+, K^-] = -\frac{1}{c}[K^-, [L_3, L^+]] = \frac{1}{c}[L^+, [K^-, L_3]] + \frac{1}{c}[L_3, [L^+, K^-]]$$

$$= \frac{1}{2c}[L^+, K^-] + \frac{1}{c}[L^3, I^-]$$

$$= \frac{1}{c} I^-$$

Comparison with Eqn. (32), gives $c = 1$.

Thus it is seen that $L^{\pm}$, $\frac{1}{2}[L^+,L^-]$ form an SU(2) algebra whose elements commute with Q_H and the algebra generated by $I^{\pm}, K^{\pm}, L^{\pm}$ is indeed an SU3 algebra.

By assuming that the operators $G_1^{\pm}=\int A_0^{\pm}d\underline{x}$, $G_1^3=\frac{1}{2}[G_1^+,G_1^-]$ (3.33), form an SU2 algebra and denoting the operators corresponding to $I^{\pm}, K^{\pm}, L^{\pm}$ by I_1, K_1, L_1 respectively, we find that if we require that

$$[G^i, G^j] = i\epsilon^{ijk}G^k$$
$$[G^i, G_1^j] = i\epsilon^{ijk}G_1^k$$
$$[G_1^i, G_1^j] = i\epsilon^{ijk}G^k \tag{3.34}$$

then the hadron charge Q_H commutes with the SU(2) × SU(2) algebra generated by $\frac{1}{2}(L^{\pm}\pm L_1^{\pm}), \frac{1}{2}(L^{\pm}\mp L_1^{\pm}), \frac{1}{2}(L^3\pm L_1^3)$. The algebra generated by $I^{\pm}, K^{\pm}, L^{\pm}, I_1^{\pm}, K_1^{\pm}, L_1^{\pm}$ is an SU3 × SU3 algebra. Furthermore, we see that if we write

$$G_1^{\pm} = \alpha_1 I_1^{\pm} + \beta_1 K_1^{\pm} \tag{3.35}$$

then as a consequence of Eqns. (34)

$$\alpha = \alpha_1 \qquad \beta = \beta_1 \tag{3.36}$$

i.e. the Cabibbo angles for the vector and axial vector currents are equal. This condition is the analogue of the presence of the projection operator $\frac{1}{2}(1+\gamma_5)$ in the case of the leptonic current.

4. Some 'Theorems' in Local Operator Theory and Their Implications.

In the preceding discussion we have concerned ourselves with the symmetry algebra of hadrons and have carefully avoided mentioning the Lie group which one would like to associate with the algebra. The reason for this is two fold. Firstly, of course, it is easier to work with the symmetry algebra rather than with the global aspects of the approximate symmetry (in case it is possible to ascribe meaning to such aspects). Secondly, it has become evident that some of the basic axioms of local field theory do not permit us to talk meaningfully of a symmetry group for the description of an approximate symmetry. On the other hand it is possible to describe the physical aspects of a 'broken' symmetry in the context of a symmetry algebra by requiring that the 'generators' acting on a physical one particle state no longer produce, in general, a superposition of one particle states, as would have been the case in the situation of an exact symmetry. Thus for a broken symmetry the matrix elements of a generator between a one particle state and a multiparticle state do not in general vanish.

To elucidate the above remarks we shall present in the following, the rough outline of a sequence of arguments culminating in the result that a generator of a broken symmetry acting on a normalisable state does not always produce a normalisable state, although it may have well defined matrix elements. Thus the generator G is not a bounded operator in the Hilbert space of physical states and it is not possible to define $\exp(i\alpha G)$ as an operator in the Hilbert space, although it is possible to give meaning to such operators by introducing a generalised Hilbert space - the so called rigged Hilbert space.

We shall present the essential ideas involved in the proof and omit for the sake of simplicity, filling in the details.

The argument is not conclusive, however. A criticism of the argument will be given subsequently.

Theorem 1:

The invariance of physical one particle states implies the invariance of the vacuum.

Consider two states $|\alpha\beta\rangle$, $|\alpha'\beta'\rangle$ each consisting of two distinct localised parts, such that the parts α, α' are at a large distance d from the parts β, β'.

Since the generator G is expressible as a space integral of a local operator, we can write in the limit $d\to\infty$ for the matrix elements of G between the states

$$\langle\alpha'\beta'|G|\alpha\beta\rangle \underset{d\to\infty}{\longrightarrow} \langle\beta'|\beta\rangle\langle\alpha'|G|\alpha\rangle + \langle\alpha'|\alpha\rangle\langle\beta'|G|\beta\rangle \tag{4.1}$$

provided $\langle 0|G|0\rangle = 0$.

The above hypothesis concerning the matrix element $\langle\alpha'\beta'|G|\alpha\beta\rangle$ can be deduced from the usual axioms of local field theory which preclude the existence of long range correlation effects. Later on we shall indicate how the above hypothesis may fail to be valid in a more general framework of ideas.

Consider the matrix element of G between a localised one particle state $|\alpha\rangle$ and the state $|\alpha'\beta'\rangle$ described above. Since $G|\alpha\rangle = \sum_n G_n|n\rangle$ (4.2) where $|n\rangle$ is a one particle state, we have $\langle\alpha'\beta'|G|\alpha\rangle = 0$ (4.3)

Now from Eqn. (1)

$$\langle\alpha'\beta'|G|\alpha\rangle \underset{d\to\infty}{\longrightarrow} \langle\alpha'|\alpha\rangle\langle\beta'|G|0\rangle + \langle\beta'|0\rangle\langle\alpha'|G|\alpha\rangle$$

$$= \langle\alpha'|\alpha\rangle\langle\beta'|G|0\rangle \tag{4.4}$$

By a suitable choice of $|\alpha\rangle$, $|\alpha'\rangle$ we can thus show that for an arbitrary state $|\beta'\rangle$

$$\langle\beta'|G|0\rangle = 0 \tag{4.5}$$

i.e. $$G|0\rangle = 0 \tag{4.5'}$$ Q.E.D.

Lemma: (Federbush and Johnson[8]).

A local operator which annihilates the vacuum is equal to zero.

The essential idea in the proof is that of crossing symmetry for a local operator which relates e.g. a matrix element $\langle ab\cdots|A|0\rangle$ to $\langle a,\cdots|A|\bar{b}\rangle$ by analytic continuation so that all matrix elements of A can be shown to vanish.

Theorem 2:

The invariance of the vacuum implies that the generator of the symmetry is time independent i.e. the symmetry is exact[9].

Suppose $$\int d\underline{x}\; J_i(\underline{x},t)\,|0\rangle = 0 \qquad i = 1,2,3 \tag{4.5a}$$

For any state $|\beta\rangle$ such that $\underline{P}|\beta\rangle = 0$ we have from Eqn. (5a)

$$\langle\beta|J_i(x)|0\rangle = 0 \tag{4.6}$$

Thus it follows that

$$\langle\beta|T_{\mu\nu}|0\rangle = \langle\beta|\partial_\mu J_\nu - \partial_\nu J_\mu|0\rangle \qquad \mu,\nu = 1,2,3,4$$

This is true for $\mu = 4$, $\nu = 1,2,3$ and $\nu = 4$, $\mu = 1,2,3$ because the time derivative simply introduce a factor E_β , the energy of the eigen state $|\beta\rangle$ and the term containing the space derivative vanishes $(\underline{P}|\beta\rangle = 0)$. It is obviously true for $\mu = \nu = 4$ and $\mu = 1,2,3$; $\nu = 1,2,3$.

Thus for an arbitrary element U_Λ of the group of Lorentz transformations

$$\langle\beta| U_\Lambda U_\Lambda^{-1} T_{\mu\nu}(0) U_\Lambda U_\Lambda^{-1} |0\rangle = 0$$

$$\Lambda_{\mu\mu'} \Lambda_{\nu\nu'} \langle\beta'| T^{\mu'\nu'}(0) |0\rangle = 0 \tag{4.8}$$

$$\text{i.e.} \quad \langle\beta'| T^{\mu'\nu'}(0) |0\rangle = 0 \tag{4.8'}$$

By a suitable choice of U_Λ we can write for an arbitrary state $|\beta'\rangle = U_\Lambda^{-1}|\beta\rangle$ where $\underline{P}|\beta\rangle = 0$. Thus for an arbitrary $|\beta'\rangle$

$$\langle\beta'| T_{\mu\nu}(x) |0\rangle = 0 \quad (4.10) \quad \text{i.e.} \quad T_{\mu\nu}(x)|0\rangle = 0 \tag{4.11}$$

We therefore have from the Lemma given above

$$T_{\mu\nu}(x) = \partial_\mu J_\nu - \partial_\nu J_\mu = 0 \tag{4.12}$$

from which it follows that

$$\frac{\partial}{\partial t} \int J_i(x)\, d\underline{x} - \int \frac{\partial}{\partial x_i} J_0(x)\, d\underline{x} = 0 \tag{4.13}$$

$$\text{i.e.} \quad \frac{\partial}{\partial t} \int J_i(x)\, d\underline{x} = 0 \tag{4.14}$$

provided

$$\int_{x_k \to +\infty} J_0(x)\, dx_i\, dx_j = \int_{x_k \to -\infty} J_0(x)\, dx_i\, dx_j .$$

It can similarly be shown under a restrictive assumption analogous to that given above:

$$\int J_0(x)\, d\underline{x}\, |0\rangle = 0 \tag{4.15}$$

implies

$$\frac{\partial}{\partial t} \int J_0(x)\, d\underline{x} = 0 \tag{4.16}$$

Theorem 3:

If the generator of a symmetry $G(t) = \int j_0(\underline{x},t)\,d\underline{x}$ exists (as a weak limit) on the vacuum states it must annihilate it[10].

The state $G(t)|0\rangle = |G\rangle$ is invariant under translations in space so that if $|G\rangle$ belongs to the Hilbert space we must have

$$\langle G|G\rangle = \langle G|G(t)|0\rangle = \int d\underline{x}\,\langle G|j_0(\underline{x},t)|0\rangle = \int d\underline{x}\,\langle G|j_0(\underline{0},t)|0\rangle \qquad (4.18)$$

finite which is possible only if it is zero. Q.E.D.

By theorem 2, we can now say that if $G(t)|0\rangle$ is defined then G is independent of time. Thus the generator of a broken symmetry can not be defined on the vacuum.

Theorem 4:

If a symmetry is broken then the generator of the symmetry can not be defined on a dense set of states belonging to the Hilbert space[11].

If $G(t)$ exists and is self adjoint then $V_\alpha(t) = e^{i\alpha G(t)}$ exists. It will be shown that this is inconsistent with broken symmetry. Let $\varphi_i(x)$ be fields transforming under operations of the Poincaré group as

$$U(a,\Lambda)\,\varphi_i(x)\,U^{-1}(a,\Lambda) = S(\Lambda^{-1})\,\varphi_i(\Lambda x + a) \qquad (4.19)$$

Suppose unitary operators $V_\alpha(t)$ exist such that

$$V_\alpha(t)\,\varphi_i(x)\,V_\alpha^+(t) = \varphi_i'(x) = D_{ij}^{(\alpha)}\,\varphi_j(x) \qquad (4.20)$$

In the situation where $V_\alpha(t)$ is independent of t the symmetry described by $V_\alpha(t)$ is exact.

Now, $[\underline{P}, V_\alpha(t)] = 0$ (4.21) $\therefore$ $\underline{P}\,V_\alpha(t)|0\rangle = 0$ (4.22)

Since the vacuum is the only translationally invariant state of finite norm, we must have $V_\alpha(t)|0\rangle = c|0\rangle$. As $V_\alpha(t)$ is unitary we have $|c|=1$. Thus it follows that the equal time Wightman functions are invariant under the unitary transformations $V_\alpha(t)$, viz.

$$\langle 0|\phi_{i_1}(\underline{x}_1,t)\cdots\phi_{i_n}(\underline{x}_n,t)|0\rangle = \langle 0|\phi'_{i_1}(\underline{x}_1,t)\cdots\phi'_{i_n}(\underline{x}_n,t)|0\rangle \tag{4.23}$$

Now, from the Hall-Wightman theorem, the first four Wightman functions are uniquely determined from their equal time values. Thus if $V_\alpha(t)$ exists the masses, the vertex functions and the elastic scattering amplitudes are all invariant under the unitary transformations $V_\alpha(t)$.

The above result holds even if a wider class of transformations is considered - viz. $\phi_i \to \phi_i'$; ϕ_i, ϕ_i' behave similarly under transformations of the Poincare group and $\phi_i'(x) = V_\alpha(t)\phi_i(\underline{x},t)V_\alpha^\dagger(t)$ The local commutativity of ϕ_i' for spacelike separation and the resulting Hall-Wightman theorem on which the above argument is based are also valid for this wider class of transformations.

5. Comments:

The above argument depends crucially on what we have called, for lack of a better term, absence of long range correlation effects which would enable us to write (4.1) and also proceed from Eqns. (4.12, 4.15) to Eqns. (4.14, 4.16) respectively. At first sight one might expect this to hold in a situation where there are no zero mass particles and therefore regard this as a valid hypothesis for a theory attempting to describe symmetries of strong interactions. A closer look at the situation, however, reveals that this need not be the case.

For example, in a situation of spontaneous breakdown of local $U(1)$ symmetry (gauge transformations of the second kind), as envisaged by Higgs[12], such effects are indeed present although there are no zero mass particles. Higgs's model comprises of a 'fundamental' two component scalar field φ coupled to a gauge field A_μ and with an appropriate self interaction term $V(\varphi^*\varphi)$. It is described by the Lagrangian density[13]

$$\mathcal{L} = -F^{\mu\nu}\partial_\nu A_\mu + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \varphi^{\mu *}(\partial_\mu + ieA_\mu)\varphi + \varphi^\mu(\partial_\mu - ieA_\mu)\varphi^* - \varphi^{\mu *}\varphi_\mu - V(\varphi^*\varphi) - G(x)\,\partial^\mu A_\mu + \tfrac{1}{2}c\,G(x)G(x) \tag{5.1}$$

where $V(|x|^2)$ has minimum for $|x| \neq 0$.

The Lagrangian (excluding terms involving $G(x)$) is invariant under gauge transformations of the second kind, viz.

$$\varphi(x) \to e^{ie\alpha(x)}\varphi(x)\,, \quad A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x) \tag{5.2}$$

The equations of motion following from Schwinger's Action Principle give the consistency condition

$$\langle 0|\,\varphi V'(\varphi^*\varphi)\,|0\rangle \tag{5.3}$$

which would determine $\langle|\varphi|0\rangle \neq 0$, if it is somehow possible to eliminate the divergences associated with products of field operators at the same point - a question we shall ignore for the present. However, we can say that if $\langle|\varphi|\rangle = \eta$ is a solution then so is $\langle|\varphi|\rangle = \eta e^{i\alpha}$, where η, α ~~is~~ are constants because of the translational invariance of the vacuum. The infinitely degenerate set of vacuum states characterised by the constant α are not transformable into each other by a proper unitary operator. In the customary language of mathematics they are said to belong to different inequivalent representations of the Hilbert space.

Following Higgs we now introduce fields R, Θ defined as follows

$$\varphi = \frac{1}{\sqrt{2}} R e^{i\Theta} \tag{5.4}$$

We shall ignore problems of operator ordering.

Define

$$\begin{aligned}
R_\mu &= \frac{1}{\sqrt{2}}\left(\varphi_\mu e^{i\Theta} + \varphi^*_\mu e^{-i\Theta}\right) \\
J_\mu &= -\frac{1}{\sqrt{2}} i \left(\varphi^*_\mu \varphi - \varphi_\mu \varphi^*\right) \\
B_i &= A_i + \frac{1}{e}\partial_i \Theta + \frac{1}{m^2}\partial_i G \qquad\qquad m = e|\eta| \\
G^0 &= A^0 + \frac{1}{m^2}\partial_k F^{0k} \\
\psi &= |\eta|\,\Theta \\
\psi^0 &= -\frac{1}{|\eta|} J^0 + \frac{1}{m}\partial_k F^{0k} \\
R &= |\eta| + R'
\end{aligned} \tag{5.5}$$

Introducing appropriate dependant field variables B_0, G^i, ψ^i we may now write

$$\begin{aligned}
\mathcal{L} = &- F^{\mu\nu}\partial_\nu B_\mu + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 B_\mu B^\mu + G^\mu \partial_\mu G + \psi^\mu \partial_\mu \psi \\
&- \frac{1}{2}\psi^\mu \psi_\mu + m\psi^\mu G_\mu + \frac{1}{2} c G^2 + \text{terms involving } R'
\end{aligned} \tag{5.6}$$

The Lagrangian thus describes a vector field B_μ associated with particles of mass $m \neq 0$ and a scalar field R' corresponding to particles of mass proportional to the second derivative of

In addition there are two fields G, ψ whose physical
$V(|x|^2)$ at $|x| = \eta$. significance will be discussed in the following.

We observe that the generator of the gauge transformations (2) is

$$G(\alpha) = \int d\underline{x} \, [\alpha(x) \, \partial^0 G(x) - G(x) \, \partial^0 \alpha(x)]$$
$$= \int d\underline{x} \, [\alpha(x)(e J^0 - \partial_k F^{0k}) - G(x) \, \partial^0 \alpha(x)] \tag{5.7}$$

In the extended operator formalism of Schwinger the physical states satisfy the gauge invariance requirement, viz.

$$G(\alpha) \, \Psi = 0 \qquad i.e. \qquad G(x) \Psi = 0 \; , \; (e J^0 - \partial_k F^{0k}) \Psi = 0 \tag{5.8}$$

In this formalism the fields $\psi(x)$, $G^0(x)$ can no longer be regarded as operators in a Hilbert space. The operators $\psi(x)$, $G^0(x)$ now act on a general functional space in which it is not possible to define a scalar product. This is a direct consequence of the conditions (5.8) which physical states have to satisfy. We can now write the conditions (5.8) in terms of the functionals representing physical states $\langle \psi', G^{0\prime}, B_i' .. | \Psi \rangle$ as

$$G \Psi = i \frac{\delta}{\delta G^{0\prime}} \langle \psi', G^{0\prime}, B_i' | \Psi \rangle = 0$$
$$-\psi^0 \Psi = i \frac{\delta}{\delta \psi} \langle \psi', G^{0\prime}, B_i' | \Psi \rangle = 0 \tag{5.8'}$$

The functionals representing physical states are thus independent of the fields ψ, G^0. The fields are thus devoid of any physical significance whatsoever and are just a manifestation of the gauge degree of freedom possessed by the Lagrangian written in terms of the basic fields A_μ, φ.

Thus in the physical situation described by the model there are no zero mass particles, although long range effects are present in the model. To demonstrate the latter statement we need only notice that the space integral of $\langle 0|[J^0(x,t), \Phi(0)]|0\rangle$ is not time independent. Indeed we may write

$$\int d\underline{x} \langle 0|[J_0(\underline{x},t), \Phi(0)]|0\rangle \approx -e \cos mt \tag{5.9}$$

in lowest order.

The details may be found in the works of Higgs[12] and Kibble[13] referred to, above.

6a. Equal Time 'Charge' Current and Current Current Commutation Relations.

We have briefly indicated above how the situation of approximate universality of meson baryon couplings and approximate mass degeneracy of meson baryon multiplets may nevertheless be characterised by exact algebraic relations between the charges introduced above. Indeed we may construct a no interaction model of a set of fermi fields in which such relations for suitably defined charges may be shown to follow from the equal time anti commutation relations for the fermi fields, although it may not be possible to assign single particle states of the fields to a representation of a Lie group. For such a model we may also derive algebraic relations which are more restrictive[14], viz.

$$\begin{aligned}
[Q_V^i(t), V_\mu^k(\underline{x},t)] &= i f_{ikl} V_\mu^l(\underline{x},t) \\
[Q_V^i(t), A_\mu^k(\underline{x},t)] &= i f_{ikl} A_\mu^l(\underline{x},t) \\
[Q_A^i(t), V_\mu^k(\underline{x},t)] &= i f_{ikl} A_\mu^l(\underline{x},t) \\
[Q_A^i(t), A_\mu^k(\underline{x},t)] &= i f_{ikl} V_\mu^l(\underline{x},t)
\end{aligned} \tag{6.1}$$

Such relations, however, may represent a higher degree of truth than the specific model from which they are derived. The ultimate test of such relations lies, of course, in the consequences that follow from them. In practice it is necessary to supplement the above relations with certain assumptions of approximate validity, to be introduced in the following and the experimental test of the relations is therefore not a very straightforward matter. However, we may safely assert that at present there is no experimental evidence contradicting the above relations, although we can not infer the same for the much more restrictive current current commutation relations - viz.

$$[V_0^i(\underline{x},t),\, V_0^k(\underline{x}',t)] = i f_{ikl}\, V_0^l(\underline{x},t)\, \delta^3(\underline{x}-\underline{x}')$$
$$[V_0^i(\underline{x},t),\, A_0^k(\underline{x}',t)] = i f_{ikl}\, A_0^l(\underline{x},t)\, \delta^3(\underline{x}-\underline{x}')$$
$$[A_0^i(\underline{x},t),\, A_0^k(\underline{x}',t)] = i f_{ikl}\, V_0^l(\underline{x},t)\, \delta^3(\underline{x}-\underline{x}') \qquad (6.2)$$

These relations are also valid in the free fermion fields model introduced above[15]. *(b)* The Goldberger-Treiman relation and the Gell-Mann-Lévy hypothesis.

It is well known that $\langle 0|V_\mu|\pi\rangle = 0$ from the requirement of invariance under G- conjugation according to which

$$G V_\mu G^{-1} = V_\mu$$
$$G|\pi\rangle = -|\pi\rangle$$
$$G|0\rangle = |0\rangle \qquad (6.3)$$

The decay of the charged pion therefore indicates that $\langle 0|A_\mu|\pi\rangle \neq 0$ From Lorentz covariance we may write

$$\langle 0|A_\mu|\pi\rangle = i f_\pi p_\mu \qquad (6.4)$$

so that $\partial^\mu A_\mu \neq 0$.

Parenthetical remark: We could have derived $\langle 0|V_\mu|\pi\rangle = 0$ alternatively from the requirement of Lorentz covariance and vector current conservation.

The numerical value of f_π appearing in equation (4) can be estimated from the experimentally known pion lifetime. Goldberger and Treiman[16] have under certain assumptions derived an important relation between f_π , the axial vector coupling constant $G_A(0)$ and the renormalised pion nucleon coupling constant $g_{NN\pi}$ which is well satisfied on substitution of the experimental values for the various coupling constants. To derive the relation we proceed by defining

$$\langle n|A_\mu|p\rangle = \frac{1}{G}\,\bar{u}_n\left[i\gamma_\mu\gamma_5\, G_A(t) + \frac{2m\,H_A(t)}{t+m_\pi^2}\,(p-n)_\mu\,\gamma_5\right]u(p) \tag{6.5}$$

We now impose the condition that in the limit $m_\pi \to 0$ $\partial^\mu A_\mu \to 0$.
This condition is made very plausible on the basis of a field theory model[17] in which a massless pseudoscalar particle is present as a consequence of the so called 'spontaneous breakdown' of invariance under the chirality transformation.

The condition thus implies

$$G_A(t) \approx H_A(t) \tag{6.6}$$

Following the standard LSZ reduction procedure we can write

$$\langle n|A_\mu|p\rangle = i\,\bar{u}_n\int e^{-in.x}\langle 0|[A_\mu(0),F(x)]|p\rangle\,\theta(x_0)\,d^4x \tag{6.7}$$

where

$$F(x) = (\gamma.\partial + m)\,\psi_n(x)$$

The integral in equation (7) is the Fourier transform of an **retarded** commutator thus suggesting that $G_A(t)$, $g_A(t) = \dfrac{2m\,H_A(t)}{t+m_\pi^2}$ (6.5') are analytic functions of $t=(n-p)^2$ in the upper half t- plane. It will now be assumed that for $g_A(t)$ we can write an unsubtracted dispersion relation.

$$g_A(t) = \frac{1}{\pi}\int dt' \, \frac{Im\, g_A(t')}{t'-t-i\epsilon} \tag{6.8}$$

Now
$$Im\langle n|A_\mu|p\rangle = \pi\sum_s \bar{u}_n \langle 0|A_\mu|s\rangle\langle s|F(0)|p\rangle\, \delta(p_{s0}+n_0-p_0) \tag{6.9}$$

The term with A_μ, F interchanged does not contribute.

Approximating the summation on the r.h.s. of Eqn. (9) by the contribution of the pion, we get

$$\frac{g_A(t)}{G} = -\frac{2f_\pi\, g_{NN\pi}\, K_{NN\pi}(t)}{t+m_\pi^2} \tag{6.10}$$

From equations (6,5',10) we finally derive

$$f_\pi = -\frac{m\; G_A(0)}{G\; g_{NN\pi}\, K_{NN\pi}(0)} \tag{6.11}$$

which is the Goldberger-Treiman relation.

In comparing the relation against experimental values it is assumed that $K_{NN\pi}(0) \simeq K_{NN\pi}(-m_\pi^2) = 1$.

From nuclear β- decay experiments $\frac{G_A(0)}{G} \simeq -1.18$.

The value of the pion nucleon coupling constant is given by

$$g^2_{NN\pi} \simeq \frac{14.5}{4\pi}$$

The value of f_π thus calculated from the relation is

$$\sqrt{2}\left(f_\pi\right)_{G.T.} \sim (0.85 \pm .06)\, m_\pi$$

On the other hand the value of f_π calculated from the experimentally observed decay rate of the pion, viz.

$$\Gamma(\pi \to \mu+\nu_\mu) = G_A^2\, \frac{f_\pi^2}{4\pi}\, m_\pi\, m_\mu^2 \left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2 = (1.84 \pm .04)\times 10^{-16}\, m_\pi$$

turns out to be

$$\sqrt{2}\,(f_\pi)_{\pi\,decay} \sim 0.95\, m_\pi$$

The two values are in quite good agreement in spite of the approximations involved which seem rather crude at first sight.

To understand the above relation in somewhat simpler terms, Gell-Mann and Levy[18] have proposed the rather natural scheme that the divergence of the axial vector current is proportional to the π meson field

$$\partial^\mu A^c_\mu = c\,\varphi^c \tag{6.12}$$

A particular consequence of this relation is now obtained by considering its matrix elements between a) the vacuum the one pion state and b) two one nucleon states. It reads

$$\frac{G^A(q^2)}{G} + \frac{q^2}{2m}\frac{g^A(q^2)}{G} \simeq -\frac{m_\pi^2}{m}\frac{1}{m_\pi^2+q^2} f_\pi\, K(q^2)\cdot g_{NN\pi} \tag{6.13}$$

where $K(q^2)$ is the pion nucleon vertex function for a virtual pion, $K(-m_\pi^2)=1$. At $q^2=0$ we have

$$f_\pi = -\frac{G_A(0)}{G}\,\frac{m}{g_{NN\pi}\,K_{NN\pi}(0)}\,. \tag{6.11}$$

which is the Goldberger-Treiman formula.

We also have

$$\lim_{q^2\to -m_\pi^2} (q^2+m_\pi^2)\, g_A(q^2) \simeq 2 f_\pi\, g_{NN\pi} \tag{6.14}$$

Attempts to describe the Gell-Mann-Levy hypothesis in the context of a field theory will be discussed in Chapter II.

7. Derivation of Relations Between Electromagnetic and Axial Vector Form Factors of Hyperons From Equal Time Commutation Relations and PCAC.

In this section several relations between the various electromagnetic and axial vector form factors of hyperons will be derived using the current algebraic methods together with those of dispersion theory. The essential objective of the following derivation is to express the magnetic moments of the hyperons and the $\Sigma\Lambda$ transition moment in terms of pion photoproduction amplitudes which are then approximated by considering the contributions of low lying baryon resonant states. The magnetic moments of the hyperons thus evaluated are found to be in agreement with their experimental values. The magnetic moments of the nucleons have been calculated by Fubini, Furlan and Rossetti[19]. Thus it is possible to make a comparison of the results with that of SU3 symmetry.

It is found that the major contributions to the magnetic moments arise from the baryon resonances forming the decuplet (with $J^P = \frac{3}{2}^+$). The contribution of the states with $J^P = \frac{3}{2}^-$ is somewhat smaller. However, the contributions of $\Lambda_\beta(1405) : J^P = \frac{1}{2}^-$ to the sum rules involving the transition moment $F_{\Sigma\Lambda}$ are very important.

We begin by briefly describing the notation used in the following. V_μ^i , A_μ^i $: i = 1,2,3$ denote respectively the isotriplets of vector and axial vector current densities. The electromagnetic current density V_μ is given by

$$V_\mu = V_\mu^3 + \frac{1}{\sqrt{3}} V_\mu^8 \qquad (7.1)$$

J_π^i is the current associated with the pion field φ^i . In the following a, b denote a one Λ or Σ particle state and since we are ignoring higher order ~~electromagnetic~~ effects we shall take $m_\Sigma = m_\Lambda$.

Define $$M^i_{\lambda\mu} = i\int d^4x\, e^{-iq\cdot x}\theta(x_0)\langle bp'|[A^i_\lambda(x), V_\mu(0)]|ap\rangle \tag{7.2}$$

Then

$$\begin{aligned} iq^\lambda M^i_{\lambda\mu} &= i\int d^4x\, e^{-iq\cdot x}\theta(x_0)\langle bp'|[\partial^\lambda A^i_\lambda(x), V_\mu(0)]|ap\rangle \\ &\quad + i\int d^4x\, e^{-iq\cdot x}\,\delta(x_0)\,\langle bp'|[A^i_0(x), V_\mu(0)]|ap\rangle \end{aligned} \tag{7.3}$$

According to the PCAC hypothesis:

$$\partial^\lambda A^i_\lambda = -f_\pi m_\pi^2 \varphi^i \tag{7.4}$$

where $$f_\pi = -\frac{G_A^{ba}(0)}{G}\frac{2m_a}{K_{ab\pi}(0)} \quad : \quad K_{ab\pi}(-m_\pi^2) = g_{ab\pi} \tag{7.4'}$$

which follows from taking the matrix element of Eqn. (4) between the states $|a\rangle$, $|b\rangle$.

We thus have from Eqns. (3) and (4)

$$\begin{aligned} &i\int d^4x\, e^{-iq\cdot x}\delta(x_0)\langle bp'|[A^i_0(x), V_\mu(0)]|ap\rangle \\ &\quad = iq^\lambda M^i_{\lambda\mu} + f_\pi \frac{m_\pi^2}{q^2+m_\pi^2} \quad R^i_\mu(\nu, \nu_1, -q^2, -k^2) \end{aligned} \tag{7.5}$$

where $$R^i_\mu(\nu,\nu_1,-q^2,-k^2) = i\int d^4x\, e^{-iq\cdot x}(\mu^2-\Box)\,\theta(x_0)\langle bp'|[\varphi^i(x), V_\mu(0)]|ap\rangle \tag{7.5'}$$

The invariants ν, ν_1, k^2 are defined as

$$\begin{aligned} m\nu &= -\tfrac{1}{2}(p+p')\cdot(p'-p+q) \\ 2m\nu_1 &= q\cdot(p'-p+q) \\ k &= p'-p+q \end{aligned} \tag{7.6}$$

If we now take the limit $q \to 0$ of Eqn. (5) we find that the left hand side of the equation reduces to

$$i\langle bp'|[\int d\underline{x}\, A^i_0(\underline{x},0),\ V_\mu(0)]|ap\rangle = i^2\epsilon_{i3j}\langle bp'|A^j_\mu(0)|ap\rangle \tag{7.7}$$

The above equation follows from the equal time commutation relations valid for the chiral SU2 × SU2 algebra of current components (Eqns. 6.1)

Thus we derive the following equation

$$i^2 \epsilon_{i3j} < b\phi' | A^j_\mu(0) | a\phi > = \lim_{q \to 0} \{ i q^\lambda M^i_{\lambda\mu} + f_\pi R^i_\mu \} \qquad (7.8)$$

To evaluate the right hand side of Eqn. (8) we write

$$R^i_\mu(\nu, \nu_1; q^2, -k^2) = i \int d^4x \, e^{-iq.x} \, \theta(x_0) < b\phi' | [J^i_\pi(x), V_\mu(0)] | a\phi >$$
$$+ i \int d^4x \, e^{-iq.x} \, \delta(x_0) < b\phi' | [\dot{\varphi}^i(x), V_\mu(0)] | a\phi >$$
$$+ i \int d^4x \, e^{-iq.x} \frac{d}{dx_0} [\delta(x_0) < b\phi' | [\varphi^i(x), V_\mu(0)] | a\phi >]$$

Integration by parts gives

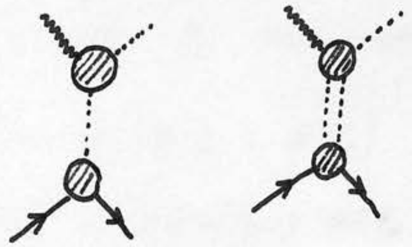
$$\int d^4x \, e^{-iq.x} \frac{d}{dx_0} [\delta(x_0) < b\phi' | [\varphi^i(x), V_\mu(0)] | a\phi >]$$
$$= i q_0 \int d^4x \, e^{-iq.x} \, \delta(x_0) < b\phi' | [\varphi^i(x), V_\mu(0)] | a\phi >$$

so that

$$R^i_\mu(t) = \lim_{q \to 0} R^i_\mu(\nu, \nu_1; q^2, -k^2) = \lim_{q \to 0} i \int d^4x \, e^{-iq.x} \, \theta(x_0) < b\phi' | [J^i_\pi(x), V_\mu(0)] | a\phi >$$
$$+ \lim_{q \to 0} i \int d^4x \, e^{-iq.x} \, \delta(x_0) < b\phi' | [\varphi^i(x), V_\mu(0)] | a\phi > \qquad (7.9)$$

The equal time commutator in the second term on the r.h.s. of Eqn. (8) is from the requirement of microcausality proportional to $\delta^3(\underline{x})$ (*) so that the entire integrand involves $\delta^4(x)$.

(*) No terms involving derivatives of $\delta^3(\underline{x})$ will be present if the first term on the r.h.s. of Eqn. (9) is covariant. See the discussion in section (3), The second term is a function of $(p-p')^2$ in the limit $q \to 0$. It is clear that the term arises from Feynman diagrams for the photoproduction amplitude which are of the type shown in the figure, i.e. diagrams resulting from the exchange of systems



with baryon number zero. To make this statement obvious, we need only point out as an example, that the three particle intermediate state $|N\pi\gamma\rangle$ does not contribute to the absorptive part of the first term on the r.h.s. of Eqn. (9).

The second term need not be considered separately. It can be incorporated in the subtraction constants occurring in the dispersion relations for the first term.

8. <u>The Invariant Amplitudes Free From Kinematic Singularities in the limit $q \to 0$</u> .

In the pioneering work of Fubini, Nambu and Watagin[20] on pion electroproduction off nucleons the invariant amplitudes for the process are introduced as follows:

The T matrix element for the process can be written as $\bar{u}_e \gamma^\mu u_e \, T_\mu$, where

$$T_\mu = \bar{u}(p')\gamma_5 \left[A M_A + B M_B + C M_C + D M_D + E M_E + F M_F \right] u(p)$$

The covariants M_A etc., are defined as

$$M_A = \tfrac{1}{2} i (\gamma_\mu \not{k} - \not{k} \gamma_\mu)$$

$$M_B = 2i\,(2m\nu_1 P_\mu + m\nu q_\mu)$$

$$M_C = 2m\nu_1 \gamma_\mu - \not{k} q_\mu$$

$$M_D = 2(-m\nu\gamma_\mu - P_\mu \not{k}) - im(\gamma_\mu \not{k} - \not{k}\gamma_\mu)$$

$$M_E = i(2m\nu_1 k_\mu - k^2 q_\mu)$$

$$M_F = \not{k} k_\mu - k^2 \gamma_\mu$$

$$\text{where} \quad P = \tfrac{1}{2}(p + p') \qquad (8.1)$$

For pions on the mass shell $q \neq 0$ the amplitudes A, B, C, D, E, F are devoid of kinematic singularities, i.e. all possible covariants that can be constructed from the set of available 'vectors' can be expressed as a linear combination of $M's$, with coefficients which are non singular. The amplitudes thus constructed are gauge invariant. This follows from the observation that $k.M_A = 0$ etc. It is now assumed on the basis of an argument in perturbation theory, that the amplitudes A, B, C, D, E, F satisfy dispersion relations in ν for $t > 0$, with the subtraction constants completely determined by the terms arising from

$$\int d^4x \, e^{-iq\cdot x} \delta(x_0) \langle b p' | [\dot{\varphi}^i(x), V_\mu(0)] | a p \rangle$$

i.e. the set of Feynman diagrams mentioned in the preceding section.

Thus, for example, we may write for $A(\nu,\nu_1,m_\pi^2,-k^2)$

$$A(\nu,\nu_1,m_\pi^2,-k^2) = A_0(\nu,\nu_1,m_\pi^2,-k^2) + \frac{1}{\pi}\int d\nu' \, Im\, A(\nu',\nu_1,m_\pi^2,-k^2)\frac{1}{\nu'-\nu-i\epsilon} \quad (8.2)$$

The dispersion relation given above may be cast into an alternative from when rewritten in terms of the isospin amplitudes A^+, A^-, A^0, defined as follows

$$A = \chi^T\{\delta_{i3}A^+ + \tfrac{1}{2}[\tau_i,\tau_3]A^- + \tau_i A^0\}\chi \quad (8.3)$$

χ is the isospinor describing the nucleon.

Thus we have

$$A^{(+,-,0)}(\nu,\nu_1,m_\pi^2,-k^2) = A_0^{(+,-,0)}(\nu,\nu_1,m_\pi^2,-k^2) + \frac{1}{\pi}\int_{\nu_0}^{\infty} d\nu' \, Im\, A^{(+,-,0)}(\nu',\nu_1,m_\pi^2,-k^2)\left[\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right]$$

$$\nu_0 = \nu_1 + m_\pi^2 + \frac{m_\pi^4}{4m^2} \quad (8.4)$$

. Similar relations hold for $B^{(+,-,0)}$ etc.

In the dispersion integral the (+) sign is applicable to the cases of $A^{(+,0)}$, $B^{(+,0)}$, $D^{(+,0)}$, $C^{(-)}$, $E^{(-)}$, $F^{(-)}$ and the (-) sign to the other cases. From the dispersion relation, it then follows that

$$\tilde{A}^{(-)}(0,\nu_1,m_\pi^2,-k^2) = \tilde{B}^{(-)}(0,\nu_1,m_\pi^2,-k^2) = \tilde{D}^{(-)}(0,\nu_1,m_\pi^2,-k^2)$$

$$= \tilde{C}^{(+,0)}(0,\nu_1,m_\pi^2,-k^2) = \tilde{E}^{(+,0)}(0,\nu_1,m_\pi^2,-k^2) = \tilde{F}^{(+,0)}(0,\nu_1,m_\pi^2,-k^2) = 0 \quad (8.5)$$

where $\tilde{A}^{(-)} = A^{(-)} - A_0^{(-)}$ etc.

It can be shown that the above set of amplitudes is not quite suitable in the limit $q \to 0$. This is because the amplitudes

B, E are singular at $\nu_1 = 0$ $\left(\nu_1 = \frac{1}{2m} q\cdot k\right)$. To see this, let us write

$$T_\mu = a\gamma_\mu + b[\gamma_\mu, \tfrac{1}{2}(\not{q}+\not{k})] + c P_\mu + \tfrac{1}{2} d P_\mu(\not{q}+\not{k})$$
$$+ \tfrac{1}{2} e (q+k)_\mu + \tfrac{1}{4} f (q+k)_\mu (\not{q}+\not{k}) + g(p'-p)_\mu + \tfrac{1}{2} h (p'-p)_\mu (\not{q}+\not{k}) \tag{8.6}$$

Then it is easy to show that

$$\begin{aligned}
A &= 2b - md \\
B &= \frac{1}{2m\nu_1}(2b + c - md) \\
C &= -\tfrac{1}{2} f - h \\
D &= -\tfrac{1}{2} d \\
E &= \frac{1}{4m\nu_1}(-e + 2g + mf - 2mh) \\
F &= \tfrac{1}{2} f - h
\end{aligned} \tag{8.7}$$

so that the amplitudes B, E are not free from kinematical singularities in the limit $q \to 0$.

It has been shown by Dennery[(21)] that a set of amplitudes A_i $(i = 1, \dots 6)$, can be introduced which are devoid of kinematic singularities at $\nu = 0,\ \nu_1 = 0$. The amplitudes A_i may be defined as follows

$$T_\mu = \bar{u}(p')\gamma_5 \sum_i A_i M_i \, u(p)$$

$$M_1 = M_A \;,\quad M_2 = \frac{4m\nu_1 - k^2}{4m\nu_1} M_B - \frac{\nu}{2\nu_1} M_E \;,\quad M_3 = M_C \;,\quad M_4 = M_D$$
$$M_5 = M_E \;,\quad M_6 = M_F \tag{8.8}$$

9. Evaluation of $\lim_{q\to 0}(iq^\lambda M^i_{\lambda\mu} + f_\pi R^i_\mu)$.

The contribution of an intermediate single particle (mass m_n) state to $\lim_{q\to 0} iq^\lambda M^i_{\lambda\mu}$ vanishes[22] for $m_n \neq m_a = m_b$. For the case $m_n = m_a = m_b$, however, the contribution is ill defined in the limit. As analogous terms also arise in the contribution of the single particle intermediate state $m_n = m_a = m_b$ to $f_\pi R^i_\mu$, so that the ill defined terms do in fact cancel in the limit, there is no difficulty in approximately evaluating $\lim_{q\to 0}(iq^\lambda M_{\lambda\mu} + f_\pi R_\mu)$ by considering the contributions of single particle states in the direct and crossed channels. The single particle states in the crossed channel correspond to multiparticle states in the direct channel.

To demonstrate the vanishing of the contribution to $\lim_{q\to 0} iq^\lambda M_{\lambda\mu}$ for $m_n \neq m_a = m_b$, and the cancellation referred to above in the case $m_n = m_a = m_b$, we write the contribution of a single particle state to $iq^\lambda M_{\lambda\mu}$ as

$$\left(iq^\lambda M_{\lambda\mu}\right)_n = iq^\lambda\Big[\frac{m_n}{E_n(\underline{p}'+\underline{q})}\frac{\langle \underline{p}' a|A_\lambda(0)|\underline{p}'+\underline{q}\, n\rangle\langle n\, \underline{p}'+\underline{q}|V_\mu(0)|\underline{p}\, b\rangle}{q_0 + E_a(\underline{p}') - E_n(\underline{p}'+\underline{q})}$$

$$- \frac{m_n}{E_n(\underline{p}-\underline{q})}\frac{\langle \underline{p}' a|V_\mu(0)|\underline{p}-\underline{q}\, n\rangle\langle \underline{p}-\underline{q}\, n|A_\lambda(0)|\underline{p}\, b\rangle}{q_0 - E_b(\underline{p}) + E_n(\underline{p}-\underline{q})}\Big] \tag{9.1}$$

where $\langle p|A_\mu(0)|p'n\rangle = i\, K_A^{an}\{(p-p')^2\}\,\overline{u}(p)\gamma_\mu\gamma_5 u(p') + \cdots$ etc.

Thus for $m_n \neq m_a = m_b$ the expression given above vanishes in the limit $q \to 0$. Now, in the limit $\underline{q}\to 0$, the denominators occurring in the above expression may be written as $p'.q$ and $p.q$

for the case $m_n = m_a = m_b$

respectively. Therefore,

$$(iq^\lambda M_{\lambda\mu})_n = -\bar{u}(p')\Big[\not{q}\gamma_5 K_A^{an}(0) \frac{m_n - i\not{p}_n}{2p'\cdot q}\gamma_\mu K_V^{bn}(k^2) - \gamma_\mu K_V^{an}(k^2)\frac{m_n - i\not{p}_n'}{2p\cdot q}\not{q}\gamma_5 K_A^{bn}(0)\Big] u(p) + O(q)$$

$$= \bar{u}(p')\Big[\Big(1 + \frac{m_n \not{q}}{p'\cdot q}\Big)\gamma_\mu K_A^{an}(0) K_V^{bn}(k^2) - \Big(1 + \frac{m_n\not{q}}{p\cdot q}\Big)\gamma_\mu K_A^{bn}(0) K_V^{an}(k^2)\Big]\gamma_5 u(p) + O(q) \qquad (9.2)$$

R_μ is well defined in the limit except for the contribution of the terms due to states (n) degenerate in mass with a or b. such a contribution is

$$(R_\mu)_n = \bar{u}(p')\Big[K_V^{bn}(k^2) K^{an\pi}(0)\frac{\not{q}}{2p'\cdot q} - K_V^{an}(k^2) K^{bn\pi}(0)\frac{\not{q}}{2p\cdot q}\Big]\gamma_\mu\gamma_5 u(p) + O(q) \qquad (9.3)$$

Thus

$$(iq^\lambda M_{\lambda\mu})_n + f_\pi (R_\mu)_n = \bar{u}(p')\Big[K_A^{an}(0) K_V^{bn}(k^2) - K_A^{bn}(0) K_V^{an}(k^2)\Big]\gamma_\mu\gamma_5 u(p) + O(q) \qquad (9.4)$$

since

$$f_\pi = -\frac{2m_n K_A^{an}(0)}{K^{an\pi}(0)} = -\frac{2m_n K_A^{bn}(0)}{K^{bn\pi}(0)}$$

In the following we shall evaluate approximately $\lim_{q\to 0}(iq^\lambda M_{\lambda\mu} + f_\pi R_\mu)$ by considering the contributions due to single particles in the various channels. On substitution of the expression for the limit in Eqn. (7.8) we derive a hierarchy of sum rules relating the various form factors for non zero momentum transfer.

10. The Sum Rules Relating the Form Factors at Non Zero Momentum Transfer.

We start from the equation

$$i^2 \epsilon_{k3\ell} \langle \Sigma p' i | A^{\ell}_{\mu}(0) | \Sigma p j \rangle = \lim_{q \to 0} \left(i q^{\lambda} M^{kij}_{\lambda\mu} + f_{\pi} R^{kij}_{\mu} \right) \qquad (7.8)$$

(where $M^{kij}_{\lambda\mu}$, R^{kij}_{μ} are defined as before with $|b\rangle = |\Sigma i\rangle, |a\rangle = |\Sigma j\rangle$

In the limit $q \to 0$ only the covariants M_1, M_6 survive. As explained above we have assumed in this derivation that the amplitudes $A_1 \dots A_6$ satisfy dispersion relations with subtraction constants determined by the ansatz given above - viz. evaluating the contributions to the amplitudes arising from the Feynman diagrams of type shown in Fig. (1).

Thus comparing the coefficients of

$$(\delta_{kj} \delta_{i3} - \delta_{ik} \delta_{j3}) \bar{u}(p') \gamma_{\mu} \gamma_5 u(p)$$
$$i(\delta_{kj} \delta_{i3} - \delta_{ik} \delta_{j3}) (p' - p)_{\mu} \bar{u}(p') \gamma_5 u(p)$$
$$i(2\delta_{ij} \delta_{3k} - \delta_{jk} \delta_{i3} - \delta_{ik} \delta_{j3}) (p' + p)_{\mu} \bar{u}(p') \gamma_5 u(p)$$
$$\epsilon_{ijk} (p' + p)_{\mu} \bar{u}(p') \gamma_5 u(p)$$
$$i(\delta_{kj} \delta_{i3} + \delta_{ik} \delta_{j3}) (p' + p)_{\mu} \bar{u}(p') \gamma_5 u(p)$$

, respectively, on both sides of equation (7.8) we obtain the following sum rules(*):

Parenthetical Remark:

In the following we shall denote for simplicity the mass of the resonance appearing in each term on the r.h.s. by the same symbol, M.

(*) The various form factors appearing in the following are defined in Appendix 2.

$$\frac{G_A(t)}{G_A(0)} = F_1^V(t) + \frac{m}{3M^2}(m+M)t\,\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H^V_{\Sigma_\delta\Sigma}(t) - \frac{m}{3M^2}(M-m)t\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H_{\Lambda_\gamma\Sigma}(t)$$

$$-\frac{m}{3M^2}(M-m)t\,\frac{K_{\Sigma'_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H^V_{\Sigma'_\beta\Sigma}(t) - \frac{m}{20M^4}(M-m)^2(M+m)\left\{(2M-m)(M+m)+t\right\}t\,H_{\Lambda^2_\alpha\Sigma}(t)\,\frac{K_{\Lambda^2_\alpha\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}$$

$$+\frac{m}{20M^4}(M+m)^2(M-m)\left\{(2M+m)(M-m)+t\right\}t\,H^V_{\Sigma_\beta\Sigma}(t)\,\frac{K_{\Sigma_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)} + \frac{m^3}{M^3}(M-m)t\,\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,h^V_{\Sigma_\delta\Sigma}(t)$$

$$+\frac{m^3}{M^3}(M+m)t\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,h_{\Lambda_\gamma\Sigma}(t) + \frac{m^3}{M^3}(M+m)t\,\frac{K_{\Sigma'_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,h^V_{\Sigma'_\beta\Sigma}(t) + \cdots\cdot$$

(10.1)

$$\frac{g_A(t)}{G_A(0)} = -\frac{2m}{t}F_1^V(t) - 2m\,\frac{K_{\Sigma\Sigma\pi}(t)}{K_{\Sigma\Sigma\pi}(0)}\,F^0_\pi(t)\left(\frac{1}{t+m_\pi^2}-\frac{1}{t}\right) - \frac{2m^2}{3M^2}(M+m)\,\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H^V_{\Sigma_\delta\Sigma}(t)$$

$$+\frac{2m^2}{3M^2}(M-m)\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H_{\Lambda_\gamma\Sigma}(t) + \frac{2m^2}{3M^2}(M-m)\,\frac{K_{\Sigma'_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H^V_{\Sigma'_\beta\Sigma}(t)$$

$$+\frac{m^2}{10M^4}(M-m)^2(M+m)\left\{(2M-m)(M+m)+t\right\}\,\frac{K_{\Lambda^2_\alpha\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H_{\Lambda^2_\alpha\Sigma}(t) + \frac{m^2}{10M^4}(M+m)^2(M-m)\left\{(2M+m)(M-m)+t\right\}$$

$$\frac{K_{\Sigma_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,H^V_{\Sigma_\beta\Sigma}(t) - \frac{2m^4}{M^3}(M-m)\,\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,h^V_{\Sigma_\delta\Sigma}(t) - \frac{2m^4}{M^3}(M+m)\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,h_{\Lambda_\gamma\Sigma}(t)$$

$$-\frac{2m^4}{M^3}(M+m)\,\frac{K_{\Sigma'_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}\,h^V_{\Sigma'_\beta\Sigma}(t) + \cdots\cdot$$

(10.2)

$$F_2^{V,S}(t) = -\frac{2m^3}{3M^2}(M+m)\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}H^{V,S}_{\Sigma_\delta\Sigma}(t) + \frac{2m^3}{3M^2}(M-m)\frac{K_{\Sigma_\rho'\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}H^{V,S}_{\Sigma_\rho'\Sigma}(t)$$

$$-\frac{m^2}{10M^4}(M^2-m^2)^2(mM+m^2+t)\frac{K_{\Sigma_\rho\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}H^{V,S}_{\Sigma_\rho\Sigma}(t) - \frac{m^2}{3M^3}(M+m)(3t-M^2+mM)\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}h^{V,S}_{\Sigma_\delta\Sigma}(t)$$

$$-\frac{m^2}{3M^3}(M-m)(3t-M^2-mM)\frac{K_{\Sigma_\rho'\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}h^{V,S}_{\Sigma_\rho'\Sigma}(t) + \cdots\cdots \tag{10.3}$$

$$F_{\Sigma\Lambda}(t) = -\frac{2m}{M-m}\frac{K_{\Sigma\Lambda_\rho\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H_{\Sigma\Lambda_\rho}(t) + \frac{2m^3}{3M^2}(M-m)\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H_{\Lambda_\gamma\Sigma}(t)$$

$$-\frac{m^2}{10M^4}(M^2-m^2)^2(t+m^2-mM)\frac{K_{\Lambda_\mu^2\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H_{\Lambda_\mu^2\Sigma}(t) - \frac{m^2}{3M^3}(M-m)(3t-M^2-mM)\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H_{\Lambda_\gamma\Sigma}(t)$$

$$+ \cdots\cdots \tag{10.4}$$

Taking $|a\rangle = |\Sigma i\rangle$, $|b\rangle = |\Lambda\rangle$ in equation (7.8) and proceeding as before we finally obtain on comparing the coefficients of

$\epsilon_{ki3}\,\bar{u}(p')\,i\gamma_\mu\gamma_5\,u(p)$, $\epsilon_{ki3}\,\bar{u}(p')(p'-p)_\mu\gamma_5\,u(p)$, $\epsilon_{ki3}\,\bar{u}(p')(p'+p)_\mu\gamma_5\,u(p)$,

$\delta_{ik}\,\bar{u}(p')\gamma_\mu\gamma_5\,u(p)$, $i\,\delta_{ik}\,\bar{u}(p')(p'+p)_\mu\gamma_5\,u(p)$ respectively

$$\frac{G_A^{\Sigma\Lambda}(t)}{G_A^{\Sigma\Lambda}(0)} = F_1^V(t) + \frac{m}{3M^2}(M+m)t\left(\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H_{\Sigma_\delta\Lambda}(t) + \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H^V_{\Sigma_\delta\Sigma}(t)\right) \tag{10.5}$$

$$- \frac{m}{3M^2}(M-m)t \left(\frac{K_{\Sigma_\rho'\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\rho'\Lambda}(t) + \frac{K_{\Sigma_\rho'\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\rho'\Sigma}(t) \right)$$

$$+ \frac{m}{20M^4}(M+m)^2(M-m)\{(2M+m)(M-m)+t\}t \left(\frac{K_{\Sigma_\rho\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\rho\Lambda}(t) + \frac{K_{\Sigma_\rho\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\rho\Sigma}(t) \right)$$

$$+ \frac{m^3}{M^3}(M-m)t \left(\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h_{\Sigma_\delta\Lambda}(t) + \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h^V_{\Sigma_\delta\Sigma}(t) \right) + \cdots\cdots$$

$$\frac{g_A^{\Sigma\Lambda}(t)}{G_A^{\Sigma\Lambda}(0)} = -\frac{2m}{t}F_1^V(t) - 2m\,\frac{K_{\Sigma\Lambda\pi}(t)}{K_{\Sigma\Lambda\pi}(0)} F_\pi^0(t)\left(\frac{1}{t+m_\pi^2} - \frac{1}{t}\right)$$

$$- \frac{2m^2}{3M^2}(M+m)\left(\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\delta\Lambda}(t) + \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\delta\Sigma}(t) \right) + \frac{2m^2}{3M^2}(M-m)\left(\frac{K_{\Sigma_\rho'\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\rho'\Lambda}(t)\right.$$

$$\left. + \frac{K_{\Sigma_\rho'\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\rho'\Sigma}(t) \right) - \frac{m^2}{10M^4}(M+m)^2(M-m)\{(2M+m)(M-m)+t\}\left(\frac{K_{\Sigma_\rho\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\rho\Lambda}(t)\right.$$

$$\left. + \frac{K_{\Sigma_\rho\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\rho\Sigma}(t) \right) - \frac{2m^4}{M^3}(M-m)\left(\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h_{\Sigma_\delta\Lambda}(t) + \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h^V_{\Sigma_\delta\Sigma}(t) \right) + \cdots$$

(10.6)

$$F_3^V(t) - \frac{G_A^{\Sigma\Sigma}(0)}{G_A^{\Sigma\Lambda}(0)} F_{\Sigma\Lambda}(t) = \frac{2m^3}{3M^2}(M+m)\left(\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\delta\Lambda}(t) - \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\delta\Sigma}(t) \right)$$

$$+ \frac{m^3}{10M^4}(M^2-m^2)^2(mM+m^2+t)\left(\frac{K_{\Sigma_\rho\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\rho\Lambda}(t) - \frac{K_{\Sigma_\rho\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H^V_{\Sigma_\rho\Sigma}(t) \right)$$

$$+ \frac{m^2}{3M^3}(M+m)(3t-M^2+mM)\left(\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h_{\Sigma_\delta\Lambda}(t) - \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h^V_{\Sigma_\delta\Sigma}(t) \right) + \cdots\cdots$$

(10.7)

$$F_1^{\Lambda}(t) - F_1^{S}(t) = \frac{m}{3M^2}(M+m)t\,\frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H^{S}_{\Sigma_\delta\Sigma}(t) + \frac{m}{3M^2}(M-m)t\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H_{\Lambda_\gamma\Lambda}(t)$$

$$- \frac{m}{20M^4}(M^2-m^2)(M+m)\left\{(m+2M)(m-M)+t\right\}t\,\frac{K_{\Sigma_\rho\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H^{S}_{\Sigma_\rho\Sigma}(t)$$

$$- \frac{m}{20M^4}(M^2-m^2)(M-m)\left\{(m-2M)(m+M)+t\right\}t\,\frac{K_{\Lambda_\omega^2\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H_{\Lambda_\omega^2\Lambda}(t)$$

$$+ \frac{m^3}{M^3}(M-m)t\,\frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,h^{S}_{\Sigma_\delta\Sigma}(t) - \frac{m^3}{M^3}(M+m)t\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,h_{\Lambda_\gamma\Lambda}(t) + \cdots\cdots$$

(10.8)

$$F_2^{S}(t) + F_2^{\Lambda}(t) = -\frac{2m}{M-m}\,\frac{K_{\Lambda_\rho\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H_{\Lambda_\rho\Lambda}(t) - \frac{2m^3}{3M^2}(M+m)\,\frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H^{S}_{\Sigma_\delta\Sigma}(t)$$

$$+ \frac{2m^3}{3M^2}(M-m)\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H_{\Lambda_\gamma\Lambda}(t) - \frac{m^2}{10M^4}(M^2-m^2)^2(t+m^2-mM)\,\frac{K_{\Lambda_\omega^2\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H_{\Lambda_\omega^2\Lambda}(t)$$

$$- \frac{m^2}{10M^4}(M^2-m^2)^2(t+m^2+mM)\,\frac{K_{\Sigma_\rho\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H^{S}_{\Sigma_\rho\Sigma}(t) - \frac{m^2}{3M^3}(M+m)(3t-M^2+mM)\,\frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,h^{S}_{\Sigma_\delta\Sigma}(t)$$

$$- \frac{m^2}{3M^3}(M-m)(3t-M^2-mM)\,\frac{K_{\Lambda_\gamma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,h_{\Lambda_\gamma\Lambda}(t) + \cdots$$

(10.9)

Finally, consider the case $|a\rangle = |\Lambda\rangle, |b\rangle = |\Lambda\rangle$. We thus obtain the following sum rule:

$$F_{\Sigma\Lambda}(t) = -\frac{2m^3}{3M^2}(M+m)\frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\delta\Lambda}(t) - \frac{m^2}{10M^4}(M^2-m^2)^2(t+m^2+mM)\frac{K_{\Sigma_\rho\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} H_{\Sigma_\rho\Lambda}(t)$$
$$-\frac{m^2}{3M^3}(M+m)(3t-M^2+mM)\frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} h_{\Sigma_\delta\Lambda}(t) + \cdots\cdot$$

(10.4')

The sum rules for Ξ are analogously found to be

$$\frac{G_A^c(t)}{G_A^c(0)} = F_1^{V(c)}(t) + \frac{m}{3M^2}(M+m)t\frac{K_{\Xi_\delta\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^V_{\Xi_\delta\Xi}(t) - \frac{m}{3M^2}(M-m)t\frac{K_{\Xi_\gamma\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^V_{\Xi_\gamma\Xi}(t)$$
$$-\frac{m}{20M^4}(M-m)^2(M+m)\left\{(2M-m)(M+m)+t\right\}t\frac{K_{\Xi_\alpha^2\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^V_{\Xi_\alpha^2\Xi}(t)$$
$$+\frac{m^3}{M^3}(M-m)t\frac{K_{\Xi_\delta\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} h^V_{\Xi_\delta\Xi}(t) + \frac{m^3}{M^3}(M+m)t\frac{K_{\Xi_\gamma\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} h^V_{\Xi_\gamma\Xi}(t) + \cdots\cdots$$

(10.10)

$$\frac{g_A^c(t)}{G_A^c(0)} = -\frac{2m}{t}F_1^{V(c)}(t) - 2m\frac{K_{\Xi\Xi\pi}(t)}{K_{\Xi\Xi\pi}(0)} F_\pi^0(t)\left(\frac{1}{t+m_\pi^2} - \frac{1}{t}\right) - \frac{2m^2}{3M^2}(M+m)\frac{K_{\Xi_\delta\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^V_{\Xi_\delta\Xi}(t)$$
$$+\frac{2m^2}{3M^3}(M-m)\frac{K_{\Xi_\gamma\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^V_{\Xi_\gamma\Xi}(t) + \frac{m^2}{10M^4}(M-m)^2(M+m)\left\{(2M-m)(M+m)+t\right\}\frac{K_{\Xi_\alpha^2\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^V_{\Xi_\alpha^2\Xi}(t)$$
$$-\frac{2m^4}{M^3}(M-m)\frac{K_{\Xi_\delta\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} h^V_{\Xi_\delta\Xi}(t) - \frac{2m^4}{M^3}(M+m)t\frac{K_{\Xi_\gamma\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} h^V_{\Xi_\gamma\Xi}(t) + \cdots\cdot$$

(10.11)

$$F_2^{V,S(c)}(t) = -\frac{2m^3}{3M^2}(M+m)\frac{K_{\Xi_\delta\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^{V,S}_{\Xi_\delta\Xi}(t) + \frac{2m^3}{3M^2}(M-m)\frac{K_{\Xi_\gamma\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^{V,S}_{\Xi_\gamma\Xi}(t)$$
$$-\frac{m^2}{10M^4}(M^2-m^2)^2(t+m^2-mM)\frac{K_{\Xi^2_\omega\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H^{V,S}_{\Xi^2_\omega\Xi}(t) - \frac{m^2}{3M^3}(M+m)(3t-M^2+mM)$$
$$\frac{K_{\Xi_\delta\Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} h^{V,S}_{\Xi_\delta\Xi}(t) + \cdots \tag{10.12}$$

11. Discussion of the Sum Rules

a) Derivation of the constraint $F_\pi^0(t)=1$.

We first notice that equations (10.1,2,5,6,10,11) together with the hypothesis of partially conserved axial vector current introduced in section (6) imply the important result that $F_\pi^0(t)=1$.

$F_\pi^0(t)$ is the extrapolation of pion electromagnetic form factor to zero mass of one of the pions. To show this we notice that PCAC and the resulting Goldberger-Treiman relation, viz.

$$f_\pi = -2m\frac{G^{ab}(0)}{G}\frac{1}{K_{ab\pi}(0)} \tag{11.1}$$

imply the following relation between

$$\frac{G_A^{ba}(t)}{G_A^{ba}(0)} + \frac{t}{2m}\frac{g_A^{ba}(t)}{G_A^{ba}(0)} = \frac{m_\pi^2}{t+m_\pi^2}\frac{K_{ab\pi}(t)}{K_{ab\pi}(0)} \tag{11.2}$$

On the other hand, multiplying equations (11.2,6,11) by and adding respectively to equations (11.1,5,10) we get

$$\frac{G_A^{ba}(t)}{G_A^{ba}(0)} + \frac{t}{2m}\frac{g_A^{ba}(t)}{G_A^{ba}(0)} = \frac{m_\pi^2}{t+m_\pi^2}\frac{K_{ab\pi}(t)}{K_{ab\pi}(0)}F_\pi^0(t) \tag{11.3}$$

so that $F_\pi^0(t)=1$. (11.4)

This a constraint which must be imposed if we are to extrapolate PCAC to zero pion mass in a consistent manner. It should be mentioned that if we had imposed PCAC in conjunction with the equal time current current commutation relations (6.2) then we would have needed the stronger condition $F_\pi(t,q^2)=1$. In particular, we would require the electromagnetic form factor of the pion $F_\pi(t) \equiv F_\pi(t,-m_\pi^2)=1$. This is too restrictive a condition and therefore we do not choose to work with the equal time current current commutation relations. The result that the equal time commutation relations, together with the extrapolated form of PCAC, implies certain non trivial dynamical constraints of the theory has also been noticed in other contexts[23].

b) Contribution to the Sum Rules Arising From States in the t-channel.

The states considered are a single pion and a single A1 meson states. They contribute only to the sum rules involving C_A and g_A. The ω and ρ meson states in the t-channel do not contribute to any of the sum rules which are obtained in the limit $q \to 0$. The possible contribution of A2 meson state has been ignored for the sake of simplicity. This does not affect the argument presented in the last section. The inclusion of higher spin states in the direct and the exchange channels is indeed irrelevant to the argument leading to the constraint $F_\pi^0(t)=1$.

12. The Decuplet $(J^P: \frac{3}{2}^+)$ Approximation and Determination of the Ratio D/F .

We now make the approximation of retaining only the contribution of $\Sigma_\delta(1385)$ to the sum rules (10.3,4',7). This approximation together with the assumption of ρ dominance of the isovector electromagnetic form factors would then enable us to determine the ratio of the symmetric (D) to the antisymmetric type (F) of SU3 invariant meson baryon coupling. The two types of couplings are defined as follows[24]:

$$
\begin{aligned}
&D \text{ type}: \quad Tr\{M(\bar{B}B + B\bar{B})\} \\
&F \text{ type}: \quad Tr\{M(\bar{B}B - B\bar{B})\}
\end{aligned}
\tag{12.1}
$$

where $M = \sum_i \varphi_i \lambda_i \; , \; B = \sum_i \psi_i \lambda_i \; , \; \bar{B} = \sum_i \psi_i^\dagger \gamma_4 \gamma_5 \lambda_i$

φ_i , ψ_i are the octets of meson and baryon fields respectively. λ_i are 3 × 3 matrices defined in reference (24).

In the approximation under consideration,

$$
F_2^V(0) = -\frac{2m^3}{3M^2}(M+m) \frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)} \left[H^V_{\Sigma_\delta\Sigma}(0) + c\, h^V_{\Sigma_\delta\Sigma}(0) \right] \tag{12.2}
$$

$$
F_{\Sigma\Lambda}(0) = -\frac{2m^3}{3M^2}(M+m) \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} \left[H_{\Sigma_\delta\Lambda}(0) + c\, h_{\Sigma_\delta\Lambda}(0) \right] \tag{12.3}
$$

$$
F_2^V(0) - \frac{G_A^{\Sigma\Sigma}}{G_A^{\Sigma\Lambda}} F_{\Sigma\Lambda}(0) = \frac{2m^3}{3M^2}(M+m) \Big[\frac{K_{\Sigma_\delta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} \left(H_{\Sigma_\delta\Lambda}(0) + c\, h_{\Sigma_\delta\Lambda}(0) \right)
- \frac{K_{\Sigma_\delta\Lambda\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} \left(H^V_{\Sigma_\delta\Sigma}(0) + c\, h^V_{\Sigma_\delta\Sigma}(0) \right) . \tag{12.4}
$$

From equations (1,2,3,4), we get

$$
\frac{H^V_{\Sigma_\delta\Sigma}(0) + c\, h^V_{\Sigma_\delta\Sigma}(0)}{H_{\Sigma_\delta\Lambda}(0) + c\, h_{\Sigma_\delta\Lambda}(0)} = -\frac{G_A^{\Sigma\Sigma}(0)}{G_A^{\Sigma\Lambda}(0)} = -\frac{K_{\Sigma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)} \approx -\frac{K_{\Sigma\Sigma\pi}(-m_\pi^2)}{K_{\Sigma\Lambda\pi}(-m_\pi^2)}
$$

$$
= -\sqrt{3}\, F/D \tag{12.5}
$$

Assuming ρ dominance of the isovector electromagnetic form factors, we have

$$\frac{H^{v}_{\Sigma_\delta\Sigma}(0) + c\, h^{v}_{\Sigma_\delta\Sigma}(0)}{H_{\Sigma_\delta\Lambda}(0) + c\, h_{\Sigma_\delta\Lambda}(0)} = \frac{K_{\Sigma_\delta\Sigma\rho}}{K_{\Sigma_\delta\Lambda\rho}} \tag{12.6}$$

In the limit of exact SU(3) for the baryon octet, baryon decuplet and vector meson decuplet we get

$$\frac{K_{\Sigma_\delta\Sigma\rho}(0)}{K_{\Sigma_\delta\Sigma\rho}(0)} \approx \frac{K_{\Sigma_\delta\Sigma\rho}(-m_\rho^2)}{K_{\Sigma_\delta\Sigma\rho}(-m_\rho^2)} = -\frac{1}{\sqrt{3}} \tag{12.7}$$

Combining this result with Eqns. (5,6), we finally obtain $D/F = 3$. The value for the ratio is in remarkable agreement with other estimates of this ratio. Martin and Wali[25] have for example carried out an extension of the Chew-Low theory of $N^*(3,3)$ to the discussion of the $P_{3/2}$ meson baryon resonances in unitary symmetry. On the basis of their work they have suggested that the observation that the resonances $P_{3/2}$ form a decuplet indicates a value for the ratio nearly equal to 3 . With $D/F = 3$ the branching ratio $\dfrac{\Gamma(\Sigma_\delta \to \Sigma\pi)}{\Gamma(\Sigma_\delta \to \Lambda\pi)}$ calculated is $\approx .067$, which is in agreement with the experimentally observed ratio $6.5 \pm 3\%$. More recently Amati et al[26] have obtained from the Cabibbo version of generalised PCAC and equal time current commutation relations a value for the ratio $D/F \sim 2.7$.

The value of D/F obtained above thus strengthens our confidence in the approximate validity of the decuplet approximation. We shall now use this approximation in the following to estimate the magnetic moments of the hyperons.

13. a) The Magnetic Moment of the Σ

The form factors at zero momentum transfer appearing in Eqn. (13.2) will be approximated as usual by the appropriate coupling constants. However, not all the coupling constants are known experimentally. We shall estimate them by using SU(3) symmetry. To form an idea of the approximate validity of SU(3) symmetry, we notice that the value of $K_{\Sigma_\delta \Sigma \pi}(-m_\pi^2)$ as determined from the Σ_δ width is very nearly equal to the SU(3) symmetry value in terms of $K_{N^* N \pi}(-m_\pi^2) = \frac{\lambda}{m_\pi}$: $\lambda = 1.81$ as calculated from the width of $N^*(3,3)$. The coupling constants $H^{V,S}_{\Sigma_\delta \Sigma}(0)$ are similarly expressed in terms of $H^V_{N^* N}(0) = \frac{C}{m_\pi}$. The analysis of pion photoproduction data by Gourdin and Salin[27] has given a value of $C \sim .345$ and also has indicated that $|H^{V,S}| >> m_\pi |h^{V,S}|$. We shall therefore ignore the term containing $h^{V,S}$. Taking $K_{NN\pi}(-m_\pi^2) = 13.5$, $K_{\Sigma\Sigma\pi}(-m_\pi^2) = \frac{2}{1+D/F} K_{NN\pi}(-m_\pi^2)$ we find that in the context of the approximations involved, the isoscalar and the isovector parts of the magnetic moments of are equal. Hence

$$\mu_{\Sigma^-} = 0$$

$$\mu_{\Sigma^0} = \frac{1}{2}\mu_{\Sigma^+} = \frac{\lambda C}{12 m_\pi^2} \frac{1+D/F}{K_{NN\pi}} \frac{m_\Sigma^2 \, m_n}{M} \left(1 + \frac{m_\Sigma}{M}\right) \tag{13.1}$$

Substituting the values of $\lambda, C, K_{NN\pi}$, $D/F = 3$ given above, we find in nuclear Bohr magnetons the value $\mu_{\Sigma^+} \sim 2.7$.

The total magnetic moments of the $\Sigma^+, \Sigma^-, \Sigma^0$ hyperon are thus found to be $3.5, -.8, 1.4$ nuclear Bohr magnetons respectively.

The value for the total magnetic moment of Σ^+ is in fair agreement with the experimental value[28], and is somewhat larger than the value in the limit of exact SU(3) which is ~ 2.79 .

b) <u>The Magnetic Moment of the Λ</u>

The determination of an approximate value of the magnetic moment of Λ will now be given in the following. It is not possible to evaluate this directly from Eqn. (10.9) in the decuplet approximation, viz.

$$F_2^S(0) + F_2^\Lambda(0) = -\frac{m_\Lambda + m_\Sigma}{m_{\Lambda_\beta} - m_\Sigma}\,\frac{K_{\Lambda_\beta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\,H_{\Lambda_\beta\Lambda}(0) - \frac{(m_\Lambda + m_\Sigma)^2 m_\Sigma (M + m_\Lambda)}{6M^2 \cdot K_{\Sigma\Lambda\pi}}\,K_{\Sigma_8\Lambda\pi}\,H^S_{\Sigma_8\Sigma}(0) \tag{13.2}$$

This is because the coupling constant $H_{\Lambda_\beta\Lambda}(0)$ is neither known experimentally nor is it possible to relate it to the known radiative decays of other baryon resonances by the SU(3) symmetry.

In the following we shall therefore derive an independent sum rule also involving the parameters of Λ_β from which the unknown quantities are then eliminated with the help of Eqn. (2). It will be assumed that Λ_β is an SU(3) singlet.

We take as our starting point the equation

$$\int [\, A_0^8(\underline{x}, 0)\,,\; V_\mu^8(0)\,]\, d\underline{x} \;=\; 0 \tag{13.3}$$

Using

$$\partial^\mu A_\mu^8(x) = c_8\, \eta(x) \tag{13.4}$$

which follows from the generalised PCAC, we may write

$$\int A_0^8(\underline{x}, 0)\, d\underline{x} \;=\; c_8 \int d^4x\, \theta(x_0)\, \eta(x) \tag{13.5}$$

so that Eqn. (3) becomes

$$\lim_{q\to 0} \int d^4x\, e^{iq\cdot x}\, \theta(x_0) \left[\eta(x) , V_\mu^8(0) \right] = 0 \tag{13.6}$$

Taking the matrix element between Λ states, we have $\lim_{q\to 0} \int d^4x\, e^{iq\cdot x}\, \theta(x_0) \langle \Lambda p' | [\eta(x), V_\mu^8(0)] | \Lambda p \rangle$. The l.h.s. of the above equation is related to the absorptive part of η photoproduction off Λ, in the limit of zero four momentum of η, and can be approximated as before in terms of the contributions of Λ and Λ_β. In this way, we get the relation

$$\mu_\Lambda = \frac{4 m_\Lambda m_p}{m_{\Lambda_\beta} - m_\Lambda} \; \frac{K_{\Lambda_\beta \Lambda \eta}(0)\; H_{\Lambda_\beta \Lambda}(0)}{K_{\Lambda\Lambda\eta}(0)} \tag{13.7}$$

where $K_{\Lambda_\beta \Lambda \eta}$, $K_{\Lambda\Lambda\eta}$ are defined as follows

$$\langle \Lambda p' | \eta(0) | \Lambda_\beta p_n \rangle = i\, K_{\Lambda_\beta \Lambda\eta}(q^2) \frac{1}{q^2 + m_\eta^2}\, \bar{u}(p') u(p_n) \tag{13.8a}$$

$$\langle \Lambda p' | \eta(0) | \Lambda p_n \rangle = i\, K_{\Lambda\Lambda\eta}(q^2) \frac{1}{q^2 + m_\eta^2}\, \bar{u}(p') \gamma_5 u(p_n) \tag{13.8b}$$

From Eqns. (2,7), we finally obtain

$$\mu_\Lambda \left[1 + \frac{m_\Lambda + m_\Sigma}{m_\Lambda} \; \frac{m_{\Lambda_\beta} - m_\Lambda}{m_{\Lambda_\beta} - m_\Sigma} \right] = -\mu_\Sigma^s - \frac{\lambda C}{8 m_\pi^2} \; \frac{(m_\Lambda + m_\Sigma)\, m_p\, m_\Sigma}{K_{NN\pi}} \cdot$$

$$(1 + F/D) \cdot \frac{1}{M} \left(1 + \frac{m_\Lambda}{M} \right)$$

$$\mu_\Sigma^s : \tfrac{1}{2} \mu_{\Sigma^+} = 1.4 \tag{13.9}$$

The value obtained for the magnetic moment of Λ is

$\mu_\Lambda = -.71$ nuclear Bohr magneton.

This is in agreement with the experimental value[(29)] of nuclear Bohr magneton.

c) The Magnetic Moments of the Ξ

From Eqn. (10.12) in the decuplet approximation we obtain

$$\mu_{\Xi}^{V,S} = \frac{2}{3} m_p \frac{m_{\Xi}^2}{M_{\Xi_8}} \left(1 + \frac{m_{\Xi}}{M_{\Xi_8}}\right) \frac{K_{\Xi_8 \Xi\pi}(0)}{K_{\Xi\Xi\pi}(0)} H_{\Xi_8 \Xi}^{V,S}(0) \tag{13.10}$$

Using the SU(3) symmetry to estimate the coupling constants appearing in the above equation, we have

$$\mu_{\Xi}^{V,S} = \frac{\lambda C}{6 m_\pi^2} \frac{m_p m_{\Xi}^2}{M_{\Xi_8}} \left(1 + \frac{m_{\Xi}}{M_{\Xi_8}}\right) \frac{1}{K_{NN\pi}} \frac{F/D + 1}{F/D - 1} \tag{13.10'}$$

Thus we get $\mu_{\Xi^-} = 0$, $\mu_{\Xi^0} = -3.1$ for the anomalous magnetic moments of the Ξ. The total magnetic moment of Ξ^- is therefore $-.71$ nuclear Bohr magneton. The SU(3) predictions for the total magnetic moments of Ξ^-, Ξ^0 are $-.88$ and -1.91 respectively. No experimental information on the magnetic moments of Ξ is at present available.

The $\Sigma\Lambda$ Transition Moment

From Eqn. (13.3), we derive with the usual approximations

$$\mu_T = \frac{\sqrt{3}}{4} \frac{\lambda C}{m_\pi^2} \frac{m_p m_\Lambda^2}{M_{\Sigma_8}} \left(1 + \frac{m_\Lambda}{M_{\Sigma_8}}\right) \frac{1}{g_{NN\pi}} \cdot (1 + F/D) \tag{13.11}$$

which gives $\mu_T = 2.0$ nuclear Bohr magneton. No experimental value for μ_T is as yet available.

14. An Alternative Derivation of the D/F Ratio

We shall now present an alternative derivation of the ratio. The value thus obtained is not far from the previous estimates. This derivation thus serves to emphasize the mutual consistency of the various assumptions employed. We shall assume as before that Λ_β constitutes an SU(3) singlet. Then we have

$$\frac{H_{\Lambda_\beta\Lambda}(0)}{H_{\Lambda_\beta\Sigma}(0)} = -\frac{1}{\sqrt{3}} \tag{14.1}$$

In the same approximation

$$F_2^{\Lambda}(0) = -\frac{2m}{M-m}\frac{K_{\Lambda_\beta\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}H_{\Lambda_\beta\Lambda}(0) \tag{14.2}$$

$$F_{\Sigma\Lambda}(0) = -\frac{2m}{M-m}\frac{K_{\Lambda_\beta\Sigma\pi}(0)}{K_{\Sigma\Sigma\pi}(0)}H_{\Lambda_\beta\Sigma}(0) \tag{14.3}$$

$$\begin{aligned}\frac{F_2^{\Lambda}(0)}{F_{\Sigma\Lambda}(0)} &= \frac{K_{\Sigma\Sigma\pi}(0)}{K_{\Sigma\Lambda\pi}(0)}\frac{H_{\Lambda_\beta\Lambda}(0)}{H_{\Lambda_\beta\Sigma}(0)} = \frac{\sqrt{3}F}{D}\left(-\frac{1}{\sqrt{3}}\right) \\ &= -\frac{F}{D}\end{aligned} \tag{14.4}$$

so that

$$D/F = -\frac{F_{\Sigma\Lambda}(0)}{F_2^{\Lambda}(0)} = \frac{2.0}{.7} \sim 2.85$$

Conclusions:

We have seen in the above discussed context, the mutual consistency of PCAC, equal time charge current commutation relations and the decuplet approximation. The estimates for the magnetic moments of Σ^+, Λ thus obtained are in fair agreement with experiment. An interesting result noticed above is the importance of the contributions of $\Lambda_\beta(1405)$ to the equations (10.4,9). Furthermore consistency can be achieved only if Λ_β belongs to an SU(3) singlet.

It is also seen that the equal time charge current commutation relations and PCAC do imply certain constraint equations, involving form factors with some of the four momenta extrapolated off the mass shell. This has been noticed in other contexts.

CHAPTER II

PROPER DEFINITION OF BILINEARS IN FIELDS AND RELATED CONSIDERATIONS

1. Introduction

It has been pointed out by Schwinger(30) and others that the usual definition of the current in quantum electrodynamics as a bilinear in the fermion field $\psi(x)$, viz.

$$j_\mu(x) = i\bar{\psi}(x)\gamma_\mu\psi(x) \tag{1.1}$$

is ambiguous, and leads to certain basic inconsistences in the theory, unless one exercises great care in evaluating expressions involving the bilinears in ψ . The 'proper' definition of the bilinears is suggested by the requirement that it enables us to avoid the inconsistencies in the theory. It should be mentioned that the definition does not in any way alter the form of the equations of motion (in terms of fields and field bilinears), as derived from Schwinger's Action Principle.

To illustrate the kind of difficulty encountered, we first give the example treated by Schwinger. Using the equal time anticommutation relations

$$\{\psi_\alpha(\underline{x},t), \bar{\psi}_\beta(\underline{x}',t)\} = (\gamma_4)_{\alpha\beta}\,\delta^3(\underline{x}-\underline{x}') \tag{1.2}$$

which follow from Schwinger's Action Principle and the requirement of symmetry under time inversion, it can be shown as a consequence of a rather straightforward calculation that

$$\begin{aligned}\left[i\bar{\psi}(x)\gamma_i\psi(x)\,,\, i\bar{\psi}(x')\gamma_0\psi(x')\right]_{x_0=x_0'} &= -\delta^3(\underline{x}-\underline{x}')\,\bar{\psi}(x)(\gamma_i\gamma_4\gamma_0-\gamma_0\gamma_4\gamma_i)\psi(x)\\ &= 0\end{aligned} \tag{1.3}$$

i.e. $\left[j_i(x),\ j_0(x') \right]_{x_0 = x_0'} = 0$ (1.4)

This leads to a contradiction as shown below.

We have $\left[\partial_i j_i(x),\ j_0(x') \right]_{x_0 = x_0'} = 0$ (1.5)

From gauge invariance of the theory, it follows by Nöether's Theorem that

$$\partial_\mu j^\mu(x) = 0 \tag{1.6}$$

so that $\partial_i j_i(x) = \partial_0 j_0(x)$ (1.6')

Thus Eqn. (5) becomes

$$\left[\partial_0 j_0(x),\ j_0(x') \right]_{x_0 = x_0'} = 0 \tag{1.7}$$

In particular, we have

$$< | [\partial_0 j_0(x),\ j_0(x')] | >_{x_0 = x_0'} = 0 \tag{1.7'}$$

The Heisenberg equations of motion following from the Action Principle give

$$[H,\ j_0(x)] = i\, \partial_0 j_0(x) \tag{1.8}$$

where H is the Hamiltonian of the system.

Thus we have $\langle 0 | [[H, j_0(x)],\ j_0(x')] | 0 \rangle_{x_0 = x_0'} = 0$ (1.8')

whence $\langle 0 | j_0(x)\ H\ j_0(x) | 0 \rangle = 0$ (1.9)

We have assumed as usual that $H|0\rangle = 0$.

Inserting a complete set of states which are eigenstates of H, we have from Eqn. (9)

$$\sum_n E_n \langle 0 | j_0(x) | n \rangle \langle n | j_0(x) | 0 \rangle = 0 \tag{1.10}$$

This is justified according to the usual ideas of field theory where H is taken to be an observable (the energy), and all the states of the system are represented by vectors in a Hilbert space. Incidentally it is not necessary to assume that $H|0\rangle = 0$ It is sufficient for the validity of the argument to assume that the vacuum is an eigenstate of H, with the lowest eigenvalue E_0. Then we obtain instead of equation (10), the following equation

$$\sum_n (E_n - E_0) \langle 0 | \dot{j}_0(x) | n \rangle \langle n | j_0(x) | 0 \rangle = 0 \tag{1.10'}$$

Now, $E_n \geq E_0$ and $\langle 0 | j_0(x) | n \rangle = \langle n | j_0(x) | 0 \rangle^*$ by the usual definition of the scalar product, according to which $\dot{j}_0(x)$ is Hermitian.

Thus we get

$$\langle n | \dot{j}_0(x) | 0 \rangle = 0 \qquad i.e. \quad \dot{j}_0(x) | 0 \rangle = 0 \tag{1.11}$$

From the Federbush-Johnson theorem, it now follows that

$$\dot{j}_0(x) = 0 \tag{1.12}$$

Thus $[j_i(x), j_0(x')]_{x_0 = x_0'} = 0 \Rightarrow j = 0$.

In a nontrivial theory, we must therefore have

$$[j_i(x), j_0(x')]_{x_0 = x_0'} \neq 0$$

To ensure that this is the case in quantum electrodynamics, Schwinger has proposed that the bilinears in ψ be interpreted as follows:

$$j_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x) = S \lim_{\epsilon \to 0} \bar{\psi}(\underline{x} + \underline{\epsilon}, x_0) \gamma_\mu \psi(\underline{x}, x_0) \tag{1.13}$$

where S stands for symmetrisation over all possible space directions of the vector $\underline{\epsilon}$.

Now, using the equal time anticommutation relations, we have

$$\left[\int d\underline{x}\, f(x)\, j_0(x) \ , \ \psi^\dagger(x'+\epsilon)\gamma_0\gamma_k\psi(x') \right]$$

$$= \left(f(x'+\underline{\epsilon}) - f(x') \right) \psi^\dagger(x'+\underline{\epsilon})\gamma_0\gamma_k\psi(x')$$

$$= \underline{\epsilon}\cdot\underline{\nabla} f(x)\, \psi^\dagger(x'+\underline{\epsilon})\gamma_0\gamma_k\psi(x') + \tfrac{1}{2}\epsilon_i\epsilon_j\, \nabla_i\nabla_j f(x)\,\psi^\dagger(x'+\underline{\epsilon})\gamma_0\gamma_k\psi(x') \tag{1.14}$$

where $f(x)$ is an arbitrary number function.

$$S \lim_{\epsilon\to 0} \underline{\epsilon}\cdot\underline{\nabla} f(x)\, \psi^\dagger(x'+\underline{\epsilon})\gamma_0\gamma_k\psi(x') = \frac{1}{3}\nabla_k f(x)\,\psi^\dagger(x'+\underline{\epsilon})\gamma_0\underline{\epsilon}\cdot\underline{\gamma}\psi(x') \tag{1.15}$$

$$S \lim_{\epsilon\to 0} \epsilon_i\epsilon_j\cdot\nabla_i\nabla_j f(x)\,\psi^\dagger(x'+\underline{\epsilon})\gamma_0\gamma_k\psi(x') = 0 \tag{1.15'}$$

Thus we have

$$\langle|[j_0(x), j_k(x')]|\rangle_{x_0=x_0'} = C\,\nabla_k\,\delta^3(\underline{x}-\underline{x}') \tag{1.16}$$

where

$$C = \frac{1}{3}\lim_{\epsilon\to 0} \mathrm{Tr}\,\gamma^0\underline{\gamma}\cdot\underline{\epsilon}\,\langle\psi(x)\psi^\dagger(x+\underline{\epsilon})\rangle \tag{1.16'}$$

Thus, if we are to have $\langle|[j_0(x), j_k(x')]|\rangle_{x_0=x_0'} \neq 0$ as is required for the consistency of the theory, we must have $C\neq 0$ so that the field product expectation value does not tend to a finite value as $\epsilon\to 0$. This is indeed well known to be true for the case of a free Dirac field, where

$$\langle 0|\psi(x)\psi^\dagger(x+\underline{\epsilon})|0\rangle \sim \frac{1}{2\pi^2}\,\frac{i\gamma_0\underline{\gamma}\cdot\underline{\epsilon}}{(\epsilon^2)^2}\,,\ \epsilon\to 0 \tag{1.17}$$

so that $C = \lim_{\epsilon\to 0}\frac{2}{3\pi^2}\frac{1}{\epsilon^2}$ is in fact infinite.

As a justification of the limiting procedure given above, we shall derive the form for $\langle|[j_0(x), j_k(x')]|\rangle$ given above (Eqn. 16) by an alternative method.

We have seen above that for any Hermitian operator A,

$$\langle|[i\partial_0 A, A]|\rangle > 0 \tag{1.18}$$

Take $A = \int E_x \, dy\, dz$, then from the equation

$$\underline{j} - \frac{\partial \underline{E}}{\partial t} = \mathrm{curl}\, \underline{H} \tag{1.19}$$

we have

$$\int j_x \, dy\, dz - \frac{\partial}{\partial t}\int E_x \, dy\, dz = \int (\mathrm{curl}\, \underline{H})_x \, dy\, dz = 0$$

so that $$\frac{\partial A}{\partial t} = \int j_x \, dy\, dz \tag{1.19'}$$

Thus we conclude from Eqn. (18), that the electric field $\underline{E}$ and the current $\underline{j}$ cannot commute. Consideration of invariance under rotations in space, together with an existence hypothesis, enables us to write

$$\langle|i[E_k(x),\, j_\ell(x')]|\rangle_{x_0 = x_0'} = C\, \delta_{k\ell} \cdot \delta^3(\underline{x} - \underline{x}') \tag{1.20}$$

Since $\underline{\nabla} \cdot \underline{E} = j_0$ we therefore get

$$\langle|i[j_0(x),\, j_\ell(x')]|\rangle_{x_0 = x_0'} = C\, \nabla_\ell \delta^3(\underline{x} - \underline{x}') \tag{1.16}$$

which is the form obtained above.

The above commutation relation (Eqn. 16), is true for any charged field. In particular, for a spinless field ϕ the current j_μ is defined as

$$j_\mu = \tfrac{1}{2}[\phi^+(-i\nabla_\mu - eA_\mu)\phi + (i\nabla_\mu - eA_\mu)\phi^+ \cdot \phi] \tag{1.21}$$

where the product $\phi^+\phi$ is properly symmetrised. From the canonical equal time commutation relations, it now follows that

$$\langle|i[j_0(x),\, j_\ell(x')]|\rangle_{x_0 = x_0'} = \langle|\phi^+\phi|\rangle\, \nabla_\ell \delta^3(\underline{x} - \underline{x}')$$

The above definition for $j_\mu(x)$ in terms of ψ (Eqn. 13) resolves only some of the difficulties associated with field bilinears at the same point. To account for nonvanishing of the commutator of $j_\mu(x)$ with the transverse electric field we find that the appropriate form for $j_\mu(x)$ is the one suggested by the requirement of gauge invariance - viz.

$$j_\mu(x) = \tfrac{1}{2}\left[\bar{\psi}(x+\underline{\epsilon})\gamma_\mu\psi(x)\, e^{i\int_x^{x+\underline{\epsilon}} d\xi^\mu A_\mu(\xi)} - \psi(x)\bar{\psi}(x+\underline{\epsilon})\gamma_\mu e^{i\int_x^{x+\epsilon} d\xi^\mu A_\mu(\xi)}\right] \tag{1.22}$$

We have indicated explicitly the appropriate antisymmetrisation of the bilinear in ψ .

2. <u>The Case of Nonconserved Currents</u>

It will now be shown that for nonconserved currents[31] $j_\mu : \partial^\mu j_\mu \neq 0$ (necessarily non trivial), we must also require that $[j_0(x), j_\ell(x')]_{x_0 = x_0'} \neq 0$.

To show this we shall first derive the Lehmann-Källen spectral representation for the vacuum expectation value of the commutator, viz.

$$\langle / [j_\mu(x)\, j_\nu(0)] / \rangle = \int ds \left\{ \rho_1(s)\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{s}\right) - \rho_0(s)\,\frac{\partial_\mu\partial_\nu}{s} \right\} \Delta(x,s) \tag{2.1}$$

where $\Delta(x,s) = \dfrac{1}{(2\pi)^3}\displaystyle\int d^4p\; \epsilon(p_0)\,\delta(p^2+s)\, e^{ip\cdot x}$.

This is easily done by considering first $\langle / j_\mu(x)\;\; j_\nu(0) / \rangle$.

$$\langle / j_\mu(x)\, j_\nu(0) / \rangle = \sum_{p_n,\alpha} \langle 0 / j_\mu(x) / p_n\alpha\rangle\langle p_n\alpha / j_\nu(0) / 0\rangle$$

$$= \frac{1}{(2\pi)^3}\int e^{ip_n\cdot x}\, \theta(p_{n0})\,\theta(-p_n^2) \sum_\alpha \langle 0 / j_\mu(0) / p_n\alpha\rangle\langle p_n\alpha / j_\nu(0) / 0\rangle\, d^4p_n \tag{2.2}$$

α together with the four momentum p_n serves to label states uniquely. The step functions $\theta(p_{n0})$, $\theta(-p_n^2)$ restrict the energy and mass respectively of the intermediate states to positive values.

From manifest Lorentz covariance we now have

$$\sum_\alpha \langle 0| j_\mu(0) |p_n\alpha\rangle \langle p_n\alpha| j_\nu(0)|0\rangle = \left(g_{\mu\nu} - \frac{p_{n\mu}p_{n\nu}}{p_n^2}\right)\rho_1(-p_n^2) - \frac{p_{n\mu}p_{n\nu}}{p_n^2}\rho_0(-p_n^2) \tag{2.3}$$

From Eqns. (2,3) we obtain

$$\langle 0| j_\mu(x)\, j_\nu(0)|0\rangle = \frac{1}{(2\pi)^3}\int e^{ip_n\cdot x}\,\theta(p_{n0})\,\theta(-p_n^2)\left\{\left(g_{\mu\nu} - \frac{p_{n\mu}p_{n\nu}}{p_n^2}\right)\rho_1(-p_n^2) - \frac{p_{n\mu}p_{n\nu}}{p_n^2}\rho_0(-p_n^2)\right\} d^4p_n \tag{2.4}$$

Expressing $\theta(-p_n^2)$ as

$$\theta(-p_n^2) = \int_0^\infty \delta(p_n^2+s)\,ds$$

we may write

$$\langle 0| j_\mu(x)\, j_\nu(0)|0\rangle = \int\left\{\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{s}\right)\rho_1(s) - \frac{\partial_\mu\partial_\nu}{s}\rho_0(s)\right\}\Delta^+(x,s)\,ds \tag{2.1'}$$

where $\Delta^\pm(x,s) = \frac{1}{(2\pi)^3}\int d^4p\,\theta(p_0)\,\delta(p^2+s)\,e^{\pm ip\cdot x}$

so that

$$\langle |[j_\mu(x), j_\nu(0)]|\rangle = \int\left\{\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{s}\right)\rho_1(s) - \frac{\partial_\mu\partial_\nu}{s}\rho_0(s)\right\}\Delta(x,s)\,ds \tag{2.1}$$

$$\Delta(x,s) = \Delta^+(x,s) - \Delta^-(x,s).$$

From Eqn. (1) we now derive the following important properties of the spectral functions $\rho_0(s)$, $\rho_1(s)$. Firstly, taking the complex conjugate of Eqn (3), interchanging the indices μ, ν we see, using the hermiticity of $j_\mu(x)$ that $\rho_0(s)$, $\rho_1(s)$

are real. Multiplying the Eqn. by $p_{n\mu} p_{n\nu}$ we get

$$-p_n^2 \rho_0(-p_n^2) \geq 0$$

so that $\rho_0(s) \geq 0 \; : \; s \geq 0$ (2.1a)

The equality sign holds only if the current is conserved. Taking the 11 component of Eqn. (3) we have

$$\left(1 - \frac{p_{n1}^2}{p_n^2}\right) \rho_1(-p_n^2) - \frac{p_{n1}^2}{p_n^2} \rho_0(-p_n^2) \geq 0 \tag{2.5}$$

p_{n1} is arbitrary so long as the condition $p_n^2 + m^2 = 0$ holds. In particular taking $p_{n1} = 0$, we derive

$$\rho_1(s) \geq 0 \tag{2.1b}$$

Thus

$$\langle 0 | [j_0(x) , j_\ell(0)] | \rangle_{x_0 = 0} = 0$$

$$\Rightarrow i \int \frac{1}{s} \left(\rho_0(s) + \rho_1(s) \right) ds \; \delta^3(\underline{x}) = 0$$

i.e. $\rho_0(s) \equiv 0$, $\rho_1(s) \equiv 0$ as $\rho_0(s), \rho_1(s) \geq 0$.

Therefore, we get from Eqn. (1), $\langle 0 | j_\mu(x) \, j_\nu(0) | 0 \rangle = 0$

In particular, $\langle 0 | j_\mu(0) \, j_\mu(0) | 0 \rangle$: no summation over the repeated index is implied.

In a field theory based on a Hilbert space with positive definite metric, the above equation implies that $j_\mu(0) | 0 \rangle = 0$. It then follows from the Federbush-Johnson Theorem that $j_\mu = 0$. Thus we have shown the necessity of the condition

$$[j_0(x) , j_\ell(x')]_{x_0 = x_0'} \neq 0$$

for a current not necessarily conserved.

3. Presence of Schwinger Terms in Commutators and Lack of Manifest Covariance of Time Ordered Products.

The important connection between the presence of the derivative terms in commutators, and lack of covariance of time ordered products, was first suggested in the context of quantum electrodynamics, by the work of K. Johnson[32]. Assuming the time ordered product is covariant, we can write to lowest order in the external field, the following formula for the induced vacuum polarisation current,

$$\langle out | j_\mu(x) | in \rangle = i \int \langle 0 | T(j_\mu(x)\ j_\nu(y)) | 0 \rangle \ A^\nu_{ext}(y)\, dy \tag{3.1}$$

This is obviously incorrect for it now follows that

$$\partial^\mu \langle out | j_\mu(x) | in \rangle = i \int \langle 0 | [j_0(x), j_\nu(y)] | 0 \rangle_{x_0 = y_0} A^\nu_{ext}(y)\, dy \neq 0 \tag{3.2}$$

as $\quad \langle 0 | [j_0(x), j_\nu(y)] | 0 \rangle_{x_0 = y_0} \neq 0$

Writing the Lehmann-Källen representation for $\langle 0 | j_\mu(x)\, j_\nu(0) | 0 \rangle$ as

$$\langle 0 | j_\mu(x) j_\nu(0) | 0 \rangle = \int (\Box g_{\mu\nu} - \partial_\mu \partial_\nu)\ \rho_1(s)\ \Delta^+(x,s)\, ds \tag{3.3}$$

we have

$$\begin{aligned} \langle 0 | [j_0(x), j_k(0)] | 0 \rangle_{x_0 = 0} &= -\partial_k \int \rho_1(s)\ \partial_0 \Delta(x,s)\Big|_{x_0 = 0} ds \\ &= i \int \rho_1(s) ds\ \delta^3(\underline{x}) \end{aligned} \tag{3.4}$$

Thus

$$\begin{aligned} \langle 0 | T(j_\mu(x)\, j_\nu(0)) | 0 \rangle &= \int (\Box g_{\mu\nu} - \partial_\mu \partial_\nu)\ \rho_1(s) \{ \theta(x_0) \Delta^+(x,s) \\ &\quad + \theta(-x_0) \Delta^-(x,s) \} ds + i\, \delta_{\mu k} \delta_{\nu k}\ \delta^4(x) \int \rho_1(s) ds \\ &= \int (\Box g_{\mu\nu} - \partial_\mu \partial_\nu)\ \rho_1(s)\ \Delta_F(x,s) + i\, \delta_{\mu k} \delta_{\nu k} \delta^4(x) \int \rho_1(s) ds \end{aligned} \tag{3.5}$$

$$\Delta_F(x,s) = \frac{1}{\ \ ^4} \int d^4q\ e^{iq\cdot x} \frac{1}{\ } \qquad \epsilon \to 0+$$

From Eqn. (4)

$$\langle 0|[j_0(x), j_0(0)]|0\rangle_{x_0=0} \neq 0 \Rightarrow \int \rho_1(s)\, ds \neq 0$$

which from Eqn. (5) implies that

$\langle 0| T(j_\mu(x)\, j_\nu(0))|0\rangle$ is non covariant.

Thus the expression for $\langle out | j_\mu(x) | in\rangle$ in Eqn. (1) has to be corrected by subtracting off the non covariant terms.

Therefore

$$\langle out|j_\mu(x)|in\rangle = i \int \langle 0| T(j_\mu(x)\, j_\nu(y))|0\rangle A^\nu_{ext}(y)\, dy$$
$$- i\, \delta_{\mu k}\, \delta_{\nu k} \int A^\nu_{ext}\, \delta^4(x-y)\, dy \int \rho_1(s)\, ds. \qquad (3.1')$$

It can now be verified that

$$\partial^\mu \langle out | j_\mu(x) | in\rangle = 0$$

An equivalent viewpoint to adopt in arriving at Eqn. (1') is now given in the following.

The conservation of the current follows from the invariance of the field equations under $\psi \to \psi + \delta\psi = \psi - i\alpha\psi$, $A_\mu \to A_\mu$, which is generated by $Q = \int dx\, j_0(x)$. (3.6)

$$i\delta\psi = [\psi(x), \alpha Q] \, .$$

In particular we have $[Q, j_\mu(x)] = 0$.

The field equations are, however, also invariant under the so called gauge transformations of the second kind, viz.

$$\psi \to \psi + \delta\psi = \psi - i\alpha(x)\psi \quad (3.7(i)) \qquad A_\mu \to A_\mu - \partial_\mu \alpha(x) \qquad (3.7(ii)$$

The current as defined above (Eqn. 1.22) is indeed invariant under 7(i) followed by 7(ii). Now transformation 7(i) is generated by $\int \alpha(x)\, j_0(x)\, dx$, therefore the change in induced by the transformation is given by

$$\delta j^k(x) = -i \int d\underline{x}' \, \alpha(x') \, [\, j^k(x) \, , j^0(x')]$$

The compensatory change induced by 7(ii) is therefore

$$\tilde{\delta} j^k(x) = - \delta j^k(x) = i \int d\underline{x}' \, \alpha(x') \, [\, j^k(x) , j^0(x')]$$
$$= \partial_k \alpha(x) \int \rho_1(s) \, ds = - \delta A_k^{ext} \int \rho_1(s) \, ds$$

Thus in the calculation of the vacuum polarisation current due to an electromagnetic field, we must take into account the explicit dependence of $j^k(x)$ on the field as given by the above equation. In this way, we arrive at the correct first order form for the vacuum polarisation current given above (Eqn. 3.1').

4. Schwinger Terms in Commutators and High Energy Behaviour of Covariant Amplitudes.

We shall now describe the connection between Schwinger terms in equal time commutators and the asymptotic behaviour of the appropriate covariant amplitudes, first suggested by Bjorken[(33)] in the context of the physically relevant class of theories.

Let the time ordered product

$$M_{\mu\nu}(q, \ldots) = i \int d^4x \, e^{iq \cdot x} \langle A | T (j_\mu^\alpha(x) \; j_\nu^\beta(0)) | B \rangle \tag{4.1}$$

and the absorptive parts

$$A^{(1)}_{\mu\nu}(q, \ldots) = \int d^4x \, e^{iq \cdot x} \langle A | j_\mu^\alpha(x) \; j_\nu^\beta(0) | B \rangle$$

$$A^{(2)}_{\mu\nu}(q, \ldots) = \int d^4x \, e^{-iq \cdot x} \langle A | j_\mu^\beta(0) \; j_\nu^\alpha(x) | B \rangle$$

(4.2)

be well defined distributions of q_0 .

Then from the theory of distributions it follows that

$$M_{\mu\nu}(q, \cdots) = \frac{1}{2\pi}\int dq_0' \left[\frac{A^{(1)}_{\mu\nu}(q_0', \underline{q}, \cdots)}{q_0' - q_0} - \frac{A^{(2)}_{\nu\mu}(q_0', -\underline{q}, \cdots)}{q_0' + q_0} \right]. \tag{4.3}$$

It is now assumed that $M_{\mu\nu} \to 0$ as $q_0 \to \infty$. This is obviously true for theories in which it is possible to approximate $A^{(1)}_{\mu\nu}$, $A^{(2)}_{\mu\nu}$ by summation over a finite number of intermediate states on the r.h.s. of Eqn. (2). Only such theories are considered as physically relevant. If this is the case, then we can change the order of integration with respect to q_0' and x_0 in the following and derive

$$M_{\mu\nu} \underset{q_0\to\infty}{\to} \frac{1}{q_0}\int \frac{dq_0'}{2\pi}\left[A^{(1)}_{\mu\nu}(q_0', \underline{q}) - A^{(2)}_{\nu\mu}(q_0', -\underline{q}) \right]$$

$$= \frac{1}{q_0}\int \frac{dq_0'}{2\pi}\left[\int d^4x\, e^{iq\cdot x} \langle A| j^{\alpha}_{\mu}(x) , j^{\beta}_{\nu}(0) |B\rangle - \int d^4x\, e^{-iq^T\cdot x} \right.$$

$$\left. \cdot \langle A| j^{\beta}_{\nu}(0)\, j^{\alpha}_{\mu}(x) |B\rangle \right.$$

$$= \frac{1}{q_0}\int d\underline{x} \langle A| [j^{\alpha}_{\mu}(x) , j^{\beta}_{\nu}(0)] |B\rangle_{x_0=0}\, e^{i\underline{q}\cdot\underline{x}} \tag{4.4}$$

so that $M_{\mu\nu} \to 0$ as $q_0 \to \infty$.

The covariant amplitude associated with the process $\beta + B \to \alpha + A$ is denoted by $\tilde{M}_{\mu\nu}(q)$. Define $M_{\mu\nu}(x)$, $\tilde{M}_{\mu\nu}(x)$ as follows

$$M_{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int e^{-iq\cdot x}\, M_{\mu\nu}(q)\, dq$$

$$\tilde{M}_{\mu\nu}(x) = \frac{1}{(2\pi)^4}\int e^{-iq\cdot x}\, \tilde{M}_{\mu\nu}(q)\, dq \tag{4.5}$$

As seen in the above example, dealing with the vacuum polarisation current (section 3), $\tilde{M}_{\mu\nu}(x)$ differs from $M_{\mu\nu}(x)$ by the Schwinger terms present in the equal time commutator which vanish for $x_0 \neq 0$. Thus

$$M_{\mu\nu}(x) = \tilde{M}_{\mu\nu}(x) \quad \text{for } x_0 \neq 0 \; . \tag{4.6}$$

From a well known theorem in the theory of distributions we therefore have

$$\tilde{M}_{\mu\nu}(x) - M_{\mu\nu}(x) = \sum_{n=1}^{N} c_n \delta^n(x_0) \tag{4.6'}$$

Taking the Fourier transform of the above equation we get

$$\tilde{M}_{\mu\nu}(q) - M_{\mu\nu}(q) = \sum_{n=1}^{N} c_n q_0^n \tag{4.7}$$

$$M_{\mu\nu}(q) \to 0 \quad \text{as} \quad q_0 \to \infty$$

therefore $$\tilde{M}_{\mu\nu}(q) \to \sum_{n=1}^{N} c_n q_0^n \; .$$

Thus it is seen that in the limit $q_0 \to \infty$ the covariant amplitude $\tilde{M}_{\mu\nu}(q)$ is non vanishing, only if there are Schwinger terms present in the equal time commutator.

5. Is the Physical Mass of a Vector Meson of Zero Bare Mass Necessarily Zero in a Manifestly Covariant Gauge Invariant Theory?

In a gauge invariant theory which is not manifestly covariant the physical mass of a vector meson of zero bare mass is in general non zero. This was first seen in the case of Higgs' model which was shown by him to be capable of meaningful description in the Coulomb gauge. Although the manifest covariance of the theory is thus forfeited, it is nevertheless possible to show that the physically significant aspects of the theory are indeed Lorentz invariant. Kibble has given a covariant description of the model by the introduction of certain rather complicated features. It has been briefly outlined in the preceding (section 1.5). In view of this, one should expect the answer to the question underlined above to be in the negative. However, we would like to present in the following, a more direct demonstration of the result, by indicating where the usual argument for the contrary becomes invalid.

Firstly, we shall present the argument given by Johnson, that the vanishing of the bare mass of a vector particle in a gauge invariant theory, implies the vanishing of the physical mass.

The vector field, coupled to a conserved current satisfies the equations,

$$\partial^\mu G_{\mu\nu} = -j_\nu + m_0^2 A_\nu \tag{5.1}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.2}$$

The canonical equal time commutation relations are

$$[A^k(x), G^{l0}(0)]_{x_0=0} = i\delta^{kl}\delta^3(\underline{x}) \tag{5.3}$$

The dependent field variables A^0, $\dot{A}^k$ are given by

$$A^0 = \frac{1}{m_0^2}[j^0 + \partial_k G^{k0}] \tag{5.4}$$

$$\dot{A}^k = G^{k0} - \partial^k A^0 = G^{k0} - \frac{1}{m_0^2}\partial^k(j^0 + \partial_l G^{l0}) \tag{5.5}$$

As the theory is manifestly covariant, we can write the spectral representation of $\langle 0| A_\mu(x)\, A_\nu(0)|0\rangle$ as

$$\langle 0| A_\mu(x)\, A_\nu(0)|0\rangle = (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) \int \frac{\rho(s)}{s}\, \Delta^+(x,s)\, ds \tag{5.6}$$

so that

$$\langle 0|[A^k(x), G^{\ell 0}(0)]|0\rangle_{x_0=0} = i\delta^{k\ell}\, \delta^3(\underline{x}) \int \rho(s) ds. \tag{5.7}$$

Comparison with Eqn. (3) gives

$$\int \rho(s)\, ds = 1 \tag{5.8}$$

From Eqns. (1,2)

$$(-\Box + m_0^2)\, A_\mu = j_\mu$$

so that from Eqn. (6)

$$\langle 0|[j_\mu(x), A_\nu(0)]|0\rangle = (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) \int (-s + m_0^2) \frac{\rho(s)}{s}\, \Delta(x,s)\, ds \tag{5.9}$$

In ref. (32) the following important assumption is made,

$\langle [j_0(x), A_\ell(0)]\rangle_{x_0=0} = 0$ (*) (5.9a) which enables us to conclude that

$$\int (-s + m_0^2) \frac{\rho(s)}{s}\, ds = 0 \tag{5.10}$$

i.e. $\dfrac{1}{m_0^2} = \displaystyle\int \frac{\rho(s)}{s}\, ds$ from Eqn. (8).

We can now see that

$\langle |[j_k(x), A_\ell(0)]|\rangle_{x_0=0}$ cannot vanish

(*)
$\langle |[j_k(x), A_\ell(0)]|\rangle_{x_0=0} = 0$, $\langle |[j_0(x), A_0(0)]|\rangle_{x_0=0}$ are always true in a covariant theory.

In fact we have

$$\langle 0|[j_k(x), \dot{A}_\ell(0)]|0\rangle_{x_0=0} = i\int (s\delta^{k\ell} - \partial^k\partial^\ell)\, \frac{\rho(s)}{s}(s-m_0^2)\,ds\ \ \delta^3(\underline{x})$$
$$= i\delta^{k\ell}\,\delta^3(\underline{x}) \int (s-m_0^2)\rho(s)\,ds$$

using Eqn. (9)

(5.11)

Now, if we require

$$\int (s-m_0^2)\rho(s)\,ds = 0$$

i.e. $$m_0^2 = \int s\rho(s)\,ds \qquad (5.12)$$

we get from Eqns. (10,12)

$$\rho(s) = C\,\delta(s-m_0^2) \qquad (5.13)$$

so that the vector particle is free and $j_\mu(x) \equiv 0$.

Alternatively, we notice that

$$\partial^k \langle|[\dot{j}_k(x), A_\ell(0)]|\rangle_{x_0=0} = 0 \Rightarrow 0 = \langle 0|[\partial_0 j_0(x), A_\ell(0)]|0\rangle$$
$$= \langle 0|[j_0, \ddot{A}_\ell(0)]|0\rangle_{x_0=0} = \langle 0|[j_0, (-\Box + m_0^2)A_\ell]|0\rangle_{x_0=0} = \langle 0|[j_0(x), j_\ell(0)]|0\rangle_{x_0=0}$$
$$\Rightarrow j = 0 .$$

We may write Eqns. (8,10) more explicitly, by writing

$$\rho(s) = Z_3\,\delta(s-m^2) + \sigma(s)$$

where m is the renormalised mass of the physical vector particle. Z_3 is the wave function renormalisation constant, and $\sigma(s)$ is the continuum contribution to the spectral function.

Then we have

$$\frac{1}{m_0^2} = \frac{Z_3}{m^2} + \int ds\, \frac{\sigma(s)}{s} \qquad (5.10')$$

$$1 = Z_3 + \int ds\, \sigma(s) \tag{5.8'}$$

In the limit $m_0 \to 0$ we deduce from Eqn. (10') the result that $m \to 0$. This completes the 'proof'.

We now propose to replace the assumption

$$\langle |[j_0(x), A_\ell(0)]| \rangle_{x_0=0} = 0 \tag{5.9a}$$

introduced above, by a somewhat different assumption, viz.

$$\langle |[j_k(x), G_{\ell 0}(0)]| \rangle_{x_0=0} = 0 \tag{5.9b}$$

From Eqn. (9), we can now derive

$$\langle |[j_k(x), G_{\ell 0}(0)]| \rangle_{x_0=0} = i \int (s - m_0^2)\, \rho(s)\, ds \;\delta_{k\ell}\, \delta^3(\underline{x}) \tag{5.14}$$

so that
$$\langle |[j_k(x), G_{\ell 0}(0)]| \rangle_{x_0=0} = 0$$

gives the condition

$$\int (s - m_0^2)\, \rho(s)\, ds = 0 \tag{5.14'}$$

i.e.

We must now require, however, that $\langle |[j_0(x), A_\ell(0)]| \rangle_{x_0=0} \neq 0$

From Eqn. (9), we have

$$\langle |[j_0(x), A_\ell(0)]| \rangle_{x_0=0} = -i\, \partial_\ell \delta^3(\underline{x}) \int \left(-1 + \frac{m_0^2}{s}\right) \rho(s)\, ds\,. \tag{5.15}$$

so that if
$$\langle |[j_0(x), A_\ell(0)]| \rangle_{x_0=0} = 0$$

we get

$$\frac{1}{m_0^2} = \int \frac{1}{s}\, \rho(s)\, ds$$

which, together with Eqn. (14') leads us to conclude as before that $j_\mu \equiv 0$.

There is no a priori reason for choosing one of the possible alternatives indicated above. In ref. (32) the following alternative is adopted,

$$\langle | [j_0(x), A_\ell(0)]_{x_0=0} | \rangle = 0 \tag{5.9a}$$

$$\langle | [j_k(x), G_{\ell 0}(0)]_{x_0=0} | \rangle = \kappa \delta_{k\ell} \, \delta^3(\underline{x}) . \tag{5.9b}$$

which is of course the correct one for quantum electrodynamics in the radiation gauge. To ensure the consistency of equations (9a,b) with the formal commutation relations of the fields, the gauge invariant limiting procedure (in the limit $m_0 \to 0$) for defining the current, is given by Eqn. (1.22).

We would like to suggest another possible alternative for a consistent theory, viz.

$$\langle | [j_k(x), G_{\ell 0}(0)]_{x_0=0} | \rangle = 0 \tag{5.9a'}$$

$$\langle | [j_0(x), A_\ell(0)]_{x_0=0} | \rangle = i \kappa' \partial_\ell \, \delta^3(\underline{x}) \tag{5.9b'}$$

The current is now defined by the following limiting procedure

$$j_\mu(x) = \lim_{\epsilon \to 0} \bar{\psi}(x+\epsilon) \gamma_\mu \, e^{-\frac{i}{m_0^2} \int_x^{x+\epsilon} \partial^\nu G_{\mu\nu} dx^\mu} \psi(x) . \tag{5.16}$$

where the antisymmetrisation of the bilinears is understood. To 'verify' (5.9a',b'), we may take $\epsilon = (\epsilon_0, \underline{0})$ and interpret $[j_0(x), A_\ell(0)]_{x_0=0}$, $[j_k(x), G_{0\ell}(0)]_{x_0=0}$ as follows:

$$[j_\mu(x), A_\ell(0)]_{x_0=0} = \lim_{\epsilon_0 \to 0} \lim_{x_0 \to 0} \bar{\psi}(x+\epsilon_0) \gamma_\mu \left(1 - \frac{i}{m_0^2} \epsilon^0 \partial^{\ell'} G_{0\ell'}(x) \right) [\psi(x), A_\ell(0)]$$

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$$+ \lim \quad \lim \quad \bar{\psi}(x+\epsilon_0) \gamma_\mu \left(\frac{-i}{} \epsilon^0 \right) \partial^{\ell'} [G_{0\ell'}(x), A_\ell(0)] \psi(x)$$

$$+ \lim_{x_0 \to 0} \lim_{x_0+\epsilon_0 \to 0} \left[\bar{\psi}(x+\epsilon), A_\ell(0)\right] \gamma_k \left(1 - \frac{i}{m_0^2}\epsilon^0 \partial^{\ell'} G_{0\ell'}(x)\right) \psi(x)$$

(5.17)

$$\left[j_k(x), G_{0\ell}(0)\right]_{x_0=0} = \lim_{\epsilon_0 \to 0} \lim_{x_0 \to 0} \bar{\psi}(x+\epsilon_0) \gamma_k \left(1 - \frac{i}{m_0^2}\epsilon^0 \partial^{\ell'} G_{0\ell'}(x)\right) \left[\psi(x), G_{0\ell}(0)\right]$$

$$+ \lim_{\epsilon_0 \to 0} \lim_{x_0 \to 0} \bar{\psi}(x+\epsilon_0) \gamma_k \left(-\frac{i}{m_0^2}\epsilon^0\right) \partial^{\ell'} \left[G_{0\ell'}(x), G_{0\ell}(0)\right] \psi(x)$$

$$+ \lim_{\epsilon_0 \to 0} \lim_{x_0+\epsilon_0 \to 0} \left[\bar{\psi}(x+\epsilon), A_\ell(0)\right] \gamma_k \left(1 - \frac{i}{m_0^2}\epsilon^0 \partial^{\ell'} G_{0\ell'}(x)\right) \psi(x) \cdot \quad (5.18)$$

The equations (5.9a',b') are thus verified if we take

$$\lim_{x_0 \to 0} \left[G_{0\ell}(x), G_{0k}(0)\right] = 0 \qquad \lim_{x_0 \to 0} \left[G_{0\ell}(x), A_k(0)\right] = i\delta_{k\ell}\, \delta^3(\underline{x}).$$

etc.

The expression on the r.h.s. of equation (5.16) (before taking the limit) is invariant under the transformations:

$$\psi(x) \to e^{i\alpha(x)} \psi(x)$$

$$\partial^\mu G_{\mu\nu} \to \partial^\mu G_{\mu\nu} + m_0^2\, \partial_\nu \alpha(x)$$

which reduce to the ordinary gauge transformations of the second kind in the limit $m_0 \to 0$.

Thus it is seen that in a theory of the above mentioned type, equation (10) may not hold. As a consequence, the argument for zero renormalised mass of the vector mesons (which is based on equation (10)) breaks down.

The possibility of constructing a gauge invariant theory of massive vector mesons, along the lines indicated above will not be examined further here.

6. On Some Field Theory Models for a Partially Conserved Axial Vector Current

We have introduced the PCAC hypothesis of Gell-Mann and Levy[18] in section 6, Chapter 1, and have subsequently used it quite satisfactorily , together with the equal time charge current commutation relations for the derivation of the hyperon magnetic moments, in terms of the appropriate coupling constants of the hyperon resonances. Similar assumptions have been made by Adler and Weisberger[34] in deducing the important relation

$$\frac{1}{G_A^2} = 1 + \frac{2m_p^2}{g_{NN\pi}^2}\,\frac{1}{\pi}\int \frac{p\,dE}{E^2}\left[\sigma^0_{\pi^- p}(E) - \sigma^0_{\pi^+ p}(E)\right] \qquad (6.1)$$

which expresses the axial vector coupling constant as an appropriate integral over the difference of the total cross sections for $\pi^- p$ and $\pi^+ p$ interactions (with the pion mass extrapolated to zero). In the above equation p , E denote respectively, the laboratory momentum and energy of the pion. From the equation, incorporating the correction for zero mass of the pion calculated on the basis of a model, Adler has obtained the value for $G_A = -1.24 \pm .03$. The experimental value for $G_A = -1.18 \pm .02$.

It is the object of the present section to describe some field theory models which have been proposed with a view to understanding the PCAC hypothesis. Gell-Mann and Levy [18] have suggested three models in which PCAC holds as a consequence of the equations of motion. None ot the models, however, is entirely satisfactory.

The 'gradient coupling' model is described by the Lagrangian density $\mathcal{L}_1(x)$:

$$\mathcal{L}_1(x) = -\bar{\psi}\left(\gamma\cdot\partial + m_0 + i f_0\, \partial^\mu \varphi\, \gamma_\mu \gamma_5\right)\psi - \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\mu_0^2\varphi^2 \tag{6.2}$$

For the sake of simplicity, we consider only the nucleon and the pion, and do not explicitly indicate the additional complications in the model, arising from the isovector character of the pion, which can be taken into account in a straightforward manner.

Consider the gauge transformation:

$$\begin{aligned} \psi(x) &\rightarrow \psi(x) \\ \varphi(x) &\rightarrow \varphi(x) - \frac{1}{f_0}\alpha \end{aligned} \tag{6.3}$$

Except for the pion mass term, the Lagrangian is invariant under the transformation. Defining the axial vector current A_μ as

$$A_\mu = \frac{\delta \mathcal{L}_1}{\delta(\partial^\mu \alpha)} = i\bar{\psi}\gamma_\mu\gamma_5\psi - \frac{1}{f_0}\partial_\mu\varphi \tag{6.4}$$

we get

$$\partial^\mu A_\mu = \partial^\mu \frac{\delta\mathcal{L}_1}{\delta(\partial^\mu\alpha)} = \frac{\delta\mathcal{L}_1}{\delta\alpha} = \frac{\mu_0^2}{f_0}\varphi \tag{6.5}$$

The following relations hold in the model

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$$\frac{Z_3 \mu_0^2}{m_\pi^2} d_\pi(0) = 1 \qquad (6.6c)$$

$$-\frac{G_A}{G} = \frac{g}{2m f_0 \sqrt{Z_3}} F_\pi(0) \qquad (6.6a) \qquad \frac{\mu_0^2}{f_0}\sqrt{Z_3} = \frac{2m}{g_r} m_\pi^2 \frac{G_A}{G} \frac{1}{d_\pi(0) F_\pi(0)} \cdot \qquad (6.6b)$$

m_π , m are the renormalised masses of the pion and the nucleon respectively. g_r is the renormalised pion nucleon coupling constant. The renormalised pion propagator and vertex function are respectively $\frac{d_\pi(k^2)}{k^2 + m_\pi^2}$ and $F_\pi(k^2)$. $d_\pi(-m_\pi^2) = F_\pi(-m_\pi^2) = 1$. $\sqrt{Z_3}$ is the pion wave function renormalisation constant.

The model, however, suffers from the serious disadvantage of divergences, which are present in every order of the perturbation expansion. These divergences cannot be removed by the renormalisation of a finite number of parameters.

Another unpleasant feature of the model is the lack of similarity between the gauge transformations that generate the vector and the axial vector currents. The model is thus incapable of accounting for the approximate symmetry between the vector and the axial vector hadron currents, which is an important feature of the weak interaction phenomena.

The σ model is another example of a field theory in which the PCAC equation holds. The model is closely related to the one proposed by Schwinger( ) for the strong interactions, and has as its characteristic feature the presence of a scalar meson σ with isospin 0 . The Lagrangian density describing the model is

$$\mathcal{L}_2 = -\bar{\psi}\left[\gamma.\partial + m_0 - g_0(\sigma + i\varphi\gamma_5)\right]\psi - \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\partial^\mu\sigma\,\partial_\mu\sigma$$

$$-\frac{1}{2}\kappa_0^2\varphi^2 - \frac{1}{2}\left(\kappa_0^2 + \frac{8\lambda_0 m_0^2}{g_0^2}\right)\sigma^2 - \lambda_0\left[(\varphi^2+\sigma^2)^2 - \frac{4m_0}{g_0}\sigma(\varphi^2+\sigma^2)\right] \tag{6.7}$$

The bilinears in ψ are to be understood as properly antisymmetrised. All the divergences occurring in the perturbation theoretic treatment of the model can now be removed by introducing a finite number of renormalisations. This is what is commonly referred to as the renormalisability of the model, and is considered to be a desirable feature.

Introducing the transformation

$$\sigma = \sigma' + \frac{m_0}{g_0} \tag{6.8}$$

the Lagrangian density $\mathcal{L}_2(x)$ may be written as

$$\mathcal{L}_2(x) = -\bar{\psi}\left[\gamma.\partial - g_0(\sigma' + i\gamma_5\varphi)\right]\psi - \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\partial^\mu\sigma'\,\partial_\mu\sigma' \tag{6.9}$$

$$-\frac{\kappa_0^2}{2}(\varphi^2 + \sigma'^2) - \lambda_0\left(\varphi^2 + \sigma'^2 - \frac{m_0^2}{g_0^2}\right)^2 - \frac{m_0\kappa_0^2}{g_0}\sigma'.$$

ignoring the addition of a constant term.

Apart from the term $\frac{m_0\kappa_0^2}{g_0}\sigma'$, the Lagrangian is invariant under the gauge transformation

$$\begin{aligned}\psi &\to (1 + i\alpha\gamma_5)\psi\\ \varphi &\to \varphi - 2\alpha\sigma'\\ \sigma' &\to \sigma' + 2\alpha\varphi\end{aligned} \tag{6.10}$$

α : infinitesimal.

In the absence of the term, linear in σ', the exact symmetry possessed by the Lagrangian, has the consequence that the renormalised mass of the fermion vanishes, and also that the physical pseudoscalar and scalar particles are degenerate in mass.

The axial vector current A_μ for the model is

$$A_\mu = - \frac{\delta \mathcal{L}_2}{\delta(\partial^\mu \alpha)} = i\bar{\psi}\gamma_\mu\gamma_5\psi - 2(\sigma\partial_\mu\phi - \phi\partial_\mu\sigma) + \frac{2m_0}{g_0}\partial_\mu\phi. \tag{6.11}$$

so that
$$\partial^\mu A_\mu = -\partial^\mu \frac{\delta \mathcal{L}_2}{\delta(\partial^\mu\alpha)} = - \frac{\delta\mathcal{L}_2}{\delta\alpha} = - \frac{2m_0 K_0^2}{g_0}\phi. \tag{6.12}$$

In addition, the Lagrangian has the exact symmetry described by the gauge transformation:

$$\begin{aligned} \psi &\to (1 + i\underline{\beta}\cdot\underline{\tau})\psi \\ \underline{\phi} &\to \underline{\phi} - 2\underline{\beta}\wedge\underline{\phi} \\ \sigma &\to \sigma \end{aligned} \tag{6.13}$$

Here we cannot avoid indicating explicitly the isovector character of the pion.

The conserved vector current associated with the transformation is

$$V_\mu = - \frac{\delta\mathcal{L}_2}{\delta(\partial^\mu\beta)} = i\bar{\psi}\gamma_\mu\underline{\tau}\psi - 2\underline{\phi}\wedge\partial_\mu\underline{\phi} \tag{6.14}$$

$$\partial^\mu V_\mu = 0$$

The two sets of gauge transformations (10,13) now bear a close relation to each other. The presence of approximate $V-A$ symmetry is indeed a satisfactory feature of the model. The two sets of transformations together form the generators of the rotation group in a four dimensional Euclidean space. constitute the basis for a representation of the group.

The relation (6c) also holds in the σ- model. However, the relations (6a,6b) cannot be derived perturbation theoretically in the model.

The model can be readily extended to include a description of the strange mesons. The spinor field ψ now represents a triplet of quarks, and pion isotriplet is replaced by the octet of pseudoscalar mesons.

We now propose to raise the following question. Is it possible to take consistently, within the framework of the model, the following equations?

$$\left[\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x), \phi(0)\right]_{x_0=0} = 0 \qquad \text{(a)} \quad (6.15)$$

$$\left[\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x), \sigma(0)\right]_{x_0=0} = 0 \qquad \text{(b)}$$

$$\left[\bar{\psi}(x)\gamma_\mu\psi(x), \phi(0)\right]_{x_0=0} = 0 \qquad \text{(c)}$$

$$\left[\bar{\psi}(x)\gamma_\mu\psi(x), \sigma(0)\right]_{x_0=0} = 0 \qquad \text{(d)}$$

If so, we would like to investigate any consequences of the above conditions for the model. We have already noticed that in the case of quantum electrodynamics,

$$\left[\bar{\psi}(x)\gamma_\mu\psi(x), A_\ell(0)\right]_{x_0=0} \neq 0.$$

and to ensure that the above condition is consistent with the equal time commutation relations of the fields, we had to introduce a suitable limiting procedure for defining $\bar{\psi}(x)\gamma_\mu\psi(x)$.

For the sake of simplicity, we shall restrict ourselves in the following, to a special case of the model, viz., that corresponding to $\lambda_0 = 0$. This restriction is of no significant consequence for the following argument.

For the case under consideration, the equations of motion and the equal time canonical commutation relations following from the Action Principle are:

$$(\gamma\cdot\partial + m_0)\psi = -g_0\left(i\gamma_5\psi\phi + \psi\sigma\right) \tag{6.16}$$

$$(\Box - \kappa_0^2)\phi = i g_0 \bar{\psi}\gamma_5\psi . \tag{6.17}$$

$$(\Box - \kappa_0^2)\sigma = g_0 \bar{\psi}\psi \tag{6.18}$$

$$\begin{aligned} &\left[\frac{\partial\phi(x)}{\partial x_0}, \phi(0)\right]_{x_0=0} = i\delta^3(\underline{x}) \\ &\{\psi(x), \bar{\psi}(0)\}_{x_0=0} = -i\gamma_0\delta^3(\underline{x}) \\ &\{\psi(x), \psi(0)\}_{x_0=0} = 0 \end{aligned} \tag{6.19}$$

All other equal time commutators between any pair of next line

$\psi(x), \bar{\psi}(x), \phi, \sigma, \frac{\partial\phi}{\partial x_0}, \frac{\partial\sigma}{\partial x_0}$ (with at most one fermi field) vanish.

We shall assume in the following, that the theory is manifestly covariant, a reasonable assumption as there are no long range forces in the model of the type encountered in theories not manifestly covariant. We also assume the existence of the various spectral representations.

Define

$$<0|[A_\mu(x), A_\nu(0)]|0> = \int\left\{\rho_1(s)\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{s}\right) - \rho_0(s)\frac{\partial_\mu\partial_\nu}{s}\right\}\Delta(x,s)\,ds \tag{6.20}$$

where $\rho_0(s),\ \rho_1(s) \geq 0$, because of the positive definiteness of the metric of the Hilbert space.

Thus

$$\langle 0|[A_\mu(x),\ \partial^\nu A_\nu(0)]|0\rangle = \int \rho_0(s)\, \partial_\mu \Delta(x,s)\, ds \,. \tag{6.12}$$

From Eqns. (12,21) we have

$$\langle 0|[A_0(x),\ \phi(0)]|0\rangle = \frac{g_0}{2m_0 K_0^2} \int \rho_0(s)\, ds \quad i\delta^3(\underline{x}) \tag{6.22}$$

From the definition of A_μ (Eqn. 11), the equal time commutation relations, and the assumption of Eqn. (15a) we deduce

$$\langle 0|[A_0(x),\ \phi(0)]_{x_0=0}|0\rangle = \left\{ \frac{2m_0}{g_0} + 2\langle 0|\sigma(0)|0\rangle \right\} i\delta^3(\underline{x}) \tag{6.23}$$

On account of the translational invariance of the vacuum, we take $\langle 0|\sigma(x)|0\rangle = \langle 0|\sigma(0)|0\rangle$.

The invariance of the vacuum under space reflection requires $\langle 0|\phi(x)|0\rangle = 0$. Comparing Eqns. (22,23) we get

$$\int \rho_0(s)\, ds = \left(\frac{2m_0 K_0}{g_0}\right)^2 + \frac{4m_0 K_0^2}{g_0} \langle 0|\sigma(0)|0\rangle \tag{6.24}$$

From Eqn. (21) we have

$$\langle 0|[\partial^\mu A_\mu(x),\ \partial^\nu A_\nu(0)]|0\rangle = \int ds\ s\, \rho_0(s)\, \Delta(x,s) \tag{6.25}$$

which reduces to

$$\langle 0|[\phi(x),\ \phi(0)]|0\rangle = \left(\frac{g_0}{2m_0 K_0^2}\right)^2 \int ds\ s\, \rho_0(s)\, \Delta(x,s) \tag{6.26}$$

because of Eqn. (12).

Eqn. (26) and the equal time commutation relation

$\left[\dfrac{\partial\phi(x)}{\partial x^0},\ \phi(0)\right]_{x_0=0} = -i\delta^3(\underline{x})$ lead to the constraint

$$\int s\,\rho_0(s)\,ds = \left(\frac{2m_0 K_0^2}{g_0}\right)^2 \tag{6.27}$$

Eqn. (18) gives

$$\begin{aligned} \langle 0|\sigma(0)|0\rangle &= -\frac{g_0}{K_0^2}\langle 0|\bar{\psi}\psi|0\rangle \\ &= -\frac{g_0}{2K_0^2}\langle 0|[\bar{\psi}_\alpha,\psi_\alpha]|0\rangle \end{aligned} \tag{6.28}$$

contraction over the repeated index is understood.

Writing the spectral representation of $\langle 0|\{\psi(x),\bar{\psi}(0)\}|0\rangle$ as

$$\langle 0|\{\psi(x),\bar{\psi}(0)\}|0\rangle = \int\{-\tau_1(s)\,\gamma.\partial + \tau_2(s)\}\,\Delta(x,s)\,ds. \tag{6.29}$$

we have from the equal time anticommutation relation for $\psi, \bar{\psi}$

$$\int \tau_1(s)\,ds = 1 \tag{6.29'}$$

From Eqn. (29) we get

$$\langle 0|[\psi_\alpha(x),\bar{\psi}_\beta(0)]|0\rangle = \int\{-\tau_1(s)\,\gamma_{\alpha\beta}.\partial + \tau_2(s)\,\delta_{\alpha\beta}\}\,\Delta^{(1)}(x,s)\,ds \tag{6.29a}$$

where we have written the indices explicitly.

Thus Eqns. (28,29a) give

$$\lim_{\epsilon\to 0}\frac{g_0}{2K_0^2}.4.\int\tau_2(s)\,\Delta^{(1)}(\epsilon,s)\,ds = \langle 0|\sigma(0)|0\rangle \tag{6.30}$$

The invariant function $\Delta^{(1)}(\epsilon,s)$ is singular on the light cone $\epsilon^2 = 0$. It can be written for small ϵ^2 as

$$\Delta^{(1)}(\epsilon,s) = \frac{1}{2\pi^2\epsilon^2} + \frac{s}{4\pi^2}\left(\ln\frac{\gamma\sqrt{s}\,|\epsilon^2|^{1/2}}{2} - \frac{1}{2}\right) + O(|\epsilon^2|^{1/2}) \tag{6.31}$$

$$\gamma = 1.781\ldots$$

We shall now prove that if the model is to be non trivial $\partial^\mu A_\mu \neq 0$, we must have $\langle 0|\sigma(0)|0\rangle$ infinite.

For if $\langle 0|\sigma(0)|0\rangle$ is to be finite we must have from Eqns. (30,31)

$$\int \tau_2(s)\, ds = 0 \tag{6.32}$$

Now, from the equation (16) we have

$$\langle 0|\{(\gamma\cdot\partial + m_0)\psi(x), \bar{\psi}(0)\}|0\rangle_{x_0=0} = -g_0\langle 0|\{i\gamma_5\psi\varphi + \psi\sigma, \bar{\psi}(0)\}|0\rangle_{x_0=0} \tag{6.33}$$

which on using the equation (29) and the equal time (anti) commutation relations reduces to

$$-g_0\langle 0|\sigma(0)|0\rangle = m_0 + \int \tau_2(s)\, ds \tag{6.33a}$$

From Eqns. (32,33a) it thus follows that

$$\langle 0|\sigma(0)|0\rangle = -\frac{m_0}{g_0} \tag{6.34}$$

Substituting this value for $\langle 0|\sigma(0)|0\rangle$ into the r.h.s. of Eqn. (24) we get

$$\int \rho_0(s)\, ds = 0 \tag{6.35}$$

Since $\rho_0(s) \geq 0$, we conclude from Eqn. (35) that $\rho_0(s) \equiv 0$.

From Eqn. (20) we now get

$$\langle 0|\partial^\mu A_\mu(0)\ \partial^\nu A_\nu(0)|0\rangle = 0 \tag{6.36}$$

In a theory with a positive definite matric the above equation implies that $\partial^\mu A_\mu|0\rangle = 0$, so that by the Lemma due to Federbush and Johnson[8]

$$\partial^\mu A_\mu = 0$$

This is unacceptable if the physical fermion mass in the model is to be non zero, and also the renormalised masses of the scalar and pseudoscalar particles are to be non degenerate. Thus we have shown that for a non trivial σ- model $(\partial^\mu A_\mu \neq 0)$ $\langle 0|\sigma(0)|0\rangle$ is infinite.

Assuming m_0, κ_0 finite, $g_0 \neq 0$ we have from Eqns. (24,27) the following results:

$\int \rho_0(s)\, ds$ is infinite

$\int s\rho_0(s)\, ds$ is finite

As $\rho_0(s) \geq 0$ we conclude from the above that $\rho_0(s)$ has a singularity at $s=0$. This means that in the spectral representation for $\langle 0|[A_\mu(x), \partial^\nu A_\nu(0)]|0\rangle$ (Eqn. 21), contributions from intermediate states of zero mass must arise to give $\rho_0(s) \neq 0$ at $s=0$.[36,37]

The argument leading to the undesirable feature of the existence of zero mass particles in the model is based on the following two assumptions:

a) m_0, κ_0 finite, $g_0 \neq 0$

b) $[\bar{\psi}\gamma_\mu\gamma_5\psi, \phi(0)]_{x_0=0} = 0$

The argument can be invalidated by letting $\kappa_0 \to \infty$. The model is likely to be meaningless in the limit $m_0 \to \infty$ as the symmetry breaking term (the fermion mass term) would become infinitely large in the limit. For $g_0 = 0$ the model is trivial.

Alternatively, we may adopt the viewpoint that next line

$$[\bar{\psi}\gamma_\mu\gamma_5\psi(x)\,,\,\phi(0)]_{x_0=0} \neq 0 \quad .$$

Indeed we find that Eqn. (22) is an identity and does not constitute an additional constraint of the system, if we define the ambiguous bilinears in fields at the same point as follows,

$$\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) = \lim_{\underline{\epsilon}\to 0} \bar{\psi}(x+\underline{\epsilon})\Big(\gamma_\mu\gamma_5 + i\,h(\epsilon)\,\underline{\epsilon}\cdot\underline{\gamma}\,\partial_\mu\phi(x)\Big)\psi(x)$$

where

$$h(\epsilon) = a + b\,\epsilon^2 \ln\epsilon + c\,\epsilon^2 \qquad \epsilon = |\vec{\epsilon}|$$

$$a = \frac{g_0}{K_0^2}\int \tau_2(s)\,ds$$

$$b = \frac{g_0}{2K_0^2}\int s\,\tau_2(s)\,ds\,.$$

$$\frac{2}{\pi^2}c = \frac{2m_0}{g_0} - \frac{g_0}{2mK_0^2}\int \rho_0(s)\,ds + \frac{g_0}{\pi^2K_0^2}\int s\,\tau_2(s)\Big(\ln\frac{\gamma\sqrt{s}}{2} - \frac{1}{2}\Big)ds + \frac{a}{2\pi^2}\int s\,\tau_1(s)\,ds \tag{6.37}$$

$$\bar{\psi}(x)\psi(x) = \lim_{\underline{\epsilon}\to 0} \bar{\psi}(x+\underline{\epsilon})\psi(x) \tag{6.38}$$

Unlike the example treated by Schwinger[30], it is not necessary here to introduce symmetrisation of the limit.

It can now be seen that the above re-definition of the bilinears has the effect of invalidating Eqn. (23), because of the additional terms which arise in the reduction of $\langle 0|[A_0(x),\,\phi(0)]|0\rangle_{x_0=0}$. These additional terms are arranged so that they exactly compensate for the terms divergent in the limit $\underline{\epsilon}\to 0$ occurring in

$$\langle 0|\sigma(0)|0\rangle = \frac{g_0}{2K_0^2}\lim_{\underline{\epsilon}\to 0}\langle 0|[\psi_\alpha(\epsilon)\,\bar{\psi}_\alpha(0)]|0\rangle$$

The reduction of $\langle 0|[A_0(x),\,\phi(0)]|0\rangle$ in Eqn. (22), now leads to an equation which is identically valid in the limit $\underline{\epsilon} \to 0$.

To show this explicitly, we proceed as follows. From the equal time commutation relations and the requirement that the bilinears in ψ are to be understood as properly anti-symmetrised, we have

$$\langle 0|[\bar{\psi}(x+\underline{\epsilon})\left(\gamma_0\gamma_5 + i h(\epsilon)\,\underline{\epsilon}\cdot\underline{\gamma}\,\partial_0\phi(x)\right)\psi(x)\,,\,\phi(0)]|0\rangle_{x_0=0}$$

$$= \frac{1}{2}\,h(\epsilon)\left(\underline{\epsilon}\cdot\underline{\gamma}\right)_{\alpha\beta}\langle 0|[\bar{\psi}_\alpha(x+\underline{\epsilon}),\psi_\beta(x)]|0\rangle\,\delta^3(\underline{x}) \tag{6.39}$$

From Eqns. (39,29a,31) we therefore obtain,

$$\langle 0|[\bar{\psi}(x+\underline{\epsilon})\left(\gamma_0\gamma_5 + i h(\epsilon)\,\underline{\epsilon}\cdot\underline{\gamma}\,\partial_0\phi(x)\right)\psi(x)\,,\,\phi(0)]|0\rangle_{x_0=0}$$

$$= 2h(\epsilon)\int\tau_1(s)\,\underline{\epsilon}\cdot\underline{\partial}\,\Delta^{(1)}(\underline{\epsilon},s)\,ds\;\delta^3(\underline{x}).$$

$$= \frac{2}{\pi^2}\,h(\epsilon)\left[-\frac{1}{\epsilon^2} + \frac{1}{4}\int s\,\tau_1(s)\,ds + O(\epsilon)\right]\delta^3(\underline{x}).$$

$$= \frac{2}{\pi^2}\left[-\frac{a}{\epsilon^2} - b\ln\epsilon - c + \frac{a}{4}\int s\,\tau_1(s)\,ds + O(\epsilon)\right]\delta^3(\underline{x}). \tag{6.40}$$

From Eqn. (29a) it follows that

$$\langle 0|[\bar{\psi}_\alpha(x+\underline{\epsilon})\,,\psi_\alpha(x)]|0\rangle = -4\int\tau_2(s)\,\Delta^{(1)}(\underline{\epsilon},s)\,ds$$

$$= -\frac{2}{\pi^2}\left[\frac{1}{\epsilon^2}\int\tau_2(s)\,ds + \frac{1}{2}\int s\,\tau_2(s)\,ds\;\ln\epsilon + \frac{1}{2}\int s\left(\ln\frac{\gamma\sqrt{s}}{2} - \frac{1}{2}\right)\tau_2(s)\,ds + O(\epsilon)\right] \tag{6.41}$$

On substitution of the expressions given above for the terms singular in the limit $\epsilon \to 0$ appearing in Eqn. (22), we can verify that the singular terms cancel out. In the limit $\epsilon \to 0$ we merely obtain an identity.

From Eqn. (21)

$$\langle 0|[A_i, \phi(0)]|0\rangle = -\frac{g_0}{2m_0 K_0^2}\int \rho_0(s)\,ds\ \partial_i \Delta(x,s) \tag{6.42}$$

so that

$$\langle 0|[A_i(x), \partial_0\phi(0)]|0\rangle = -\frac{i g_0}{2m_0 K_0^2}\int \rho_0(s)\,ds\ \partial_i \delta^3(\underline{x}) \tag{6.43}$$

It can similarly be verified that with the above definition of the bilinears, the singular terms occurring in the reduction of $\langle 0|[A_i(x), \partial_0\phi(0)]|0\rangle_{x_0=0}$ also cancel out, and the Eqn. (43) is identically true in the limit $\epsilon \to 0$.

The Eqns. (6.15b,c,d) may be maintained in the model without leading to physically undesirable results. Indeed we may maintain the point of view that the (badly defined) Lagrangian and the action principle do not completely specify a field theory. It is not meaningful to ask whether equations such as Eqn. (15) hold in the model, for which only the ambiguous Lagrangian density is given. It seems that within the framework of present ideas of field theory, such equations must be separately given to determine the theory. A restrictive principle, such as gauge invariance of the second kind is thus precisely what is required for removal of the above mentioned arbitrariness, in a field theory. Even then it is not obvious that the theory is uniquely determined as we have suggested in the foregoing discussion (section 5).

CHAPTER III

SUPERCONVERGENCE RELATIONS

1. Introduction

Attempts have been made to correlate high energy behaviour with information about low lying resonances through the analyticity in the complex angular momentum plane, which was first introduced by Regge[38]. Although the idea has fallen into some disrepute lately through its inability to describe correctly the features of the so called 'high energy scattering', interest in the idea has been revived recently through the realisation that some of the results of equal time current commutation relations, are merely expressions of appropriate asymptotic behaviour of certain amplitudes as suggested by 'Reggeology.

As has been pointed out by Alfaro et al.[39], the equal time commutation relations of currents α, β together with the assumption of covariance of $\int e^{iq.x}\theta(x_0)\langle p'|[j_\mu^\alpha(x), j_\nu^\beta(0)]|p\rangle d^4x$ give us relations of the form

$$\int Im\, A^{\alpha\beta}(s,t,q^2,k^2)\, ds = C^{\alpha\beta\gamma} F^\gamma(t) \tag{1.1}$$

where $A^{\alpha\beta}$ is an invariant amplitude occurring in the decomposition of $\int e^{iq.x}\theta(x_0)\langle p'|[j_\mu^\alpha(x), j_\nu^\beta(0)]|p\rangle d^4x$. q, k are the momenta associated with the currents α, β , and s, t, k^2 are certain invariants constructed from p, p' and q , defined in the following. $F^\gamma(t)$ is a form factor associated with $\langle p'|j_\mu^\gamma|p\rangle$.

From relations of the type given above, under suitable assumptions of pole dominance of currents, i.e.

$$\langle A | j^{\alpha} | B \rangle \simeq \sum_{i(\alpha)} \langle 0 | j^{\alpha} | \varphi_{i(\alpha)} \rangle \langle A | \varphi_{i(\alpha)} | B \rangle \tag{1.2}$$

(the summation is over a finite set of particle fields), we obtain

$$\int \mathrm{Im}\, A^{i(\alpha)\, j(\beta)}_{\text{phys.}}(s,t)\, ds = 0 \tag{1.3}$$

It was realised by Alfaro et al[39] and independently by Solov'ev[40], that such relations do in fact follow from appropriate high energy behaviour of the amplitudes, without recourse to the detailed formalism of equal time commutation relations and pole dominance of currents.

2. Derivation of Superconvergence Sum Rules From the Equal Time Current Commutation Relations, and Covariance of the Retarded Product

Define

$$T_{\mu\nu} = i \int e^{iq\cdot x}\, \theta(x_0) \langle p' | [j^{\alpha}_{\mu}(x), j^{\beta}_{\nu}(0)] | p \rangle\, d^4x \tag{2.1}$$

$$t_{\mu\nu} = \frac{1}{2} \int e^{iq\cdot x} \langle p' | [j^{\alpha}_{\mu}(x), j^{\beta}_{\nu}(0)] | p \rangle\, d^4x \tag{2.2}$$

where $|p\rangle$, $|p'\rangle$ are one particle states of spin zero.

Then

$$-iq^{\mu} T_{\mu\nu} = -i \int \langle p' | [j^{\alpha}_{0}(x), j^{\beta}_{\nu}(0)]_{x_0=0} | p \rangle\, e^{i\underline{q}\cdot\underline{x}}\, d\underline{x} + i \int e^{iq\cdot x}\, \theta(x_0) \langle p' | [\partial^{\mu} j^{\alpha}_{\mu}(x), j^{\beta}_{\nu}(0)] | p \rangle\, d^4x \tag{2.3}$$

$$q^{\mu} t_{\mu\nu} = i \int e^{iq\cdot x} \langle p' | [\partial^{\mu} j^{\alpha}_{\mu}(x), j^{\beta}_{\nu}(0)] | p \rangle\, d^4x \tag{2.4}$$

From the theory of distributions,

$$T_{\mu\nu}(q_0, \underline{q}) = \frac{1}{\pi} \int \frac{t_{\mu\nu}(q_0', \underline{q})}{q_0' - q_0 - i\epsilon}\, dq_0' . \tag{2.5a}$$

and $$W_\mu = \frac{1}{\pi}\int \frac{\omega_\mu(q_0', \underline{q})}{q_0' - q_0 - i\epsilon}\, dq_0' \qquad (2.5b)$$

where

$$W_\mu = \int e^{iq.x}\, \theta(x_0) \langle p'|[\partial^\lambda j_\lambda^\alpha(x), j_\mu^\beta(0)]|p\rangle\, d^4x$$

$$\omega_\mu = -\frac{i}{2}\int e^{iq.x} \langle p'|[\partial^\lambda j_\lambda^\alpha(x), j_\mu^\beta(0)]|p\rangle\, d^4x$$

The assumed covariance of $T_{\mu\nu}$ and W_μ enables us to write

$$T_{\mu\nu} = A\, P_\mu P_\nu + B_1 P_\mu Q_\nu + B_2 P_\mu \Delta_\nu + B_3 P_\nu Q_\mu + B_4 P_\nu \Delta_\mu + C_1 Q_\mu Q_\nu + C_2 Q_\mu \Delta_\nu + C_3 Q_\nu \Delta_\mu + C_4 \Delta_\mu \Delta_\nu + C_5 g_{\mu\nu} \qquad (2.6)$$

$$W_\mu = L P_\mu + N_1 Q_\mu + N_2 \Delta_\mu \qquad (2.6')$$

and $t_{\mu\nu} = a\, P_\mu P_\nu + b_1 P_\mu Q_\nu + \dots$, $\omega_\mu = l P_\mu + n_1 Q_\mu + n_2 \Delta_\mu$

where $P = \frac{1}{2}(p+p')$, $\Delta = p - p'$, $Q = \frac{1}{2}(p'-p) + q$

Define the invariants ν, t as $\nu = P.Q$, $t = \Delta^2$.

Then Eqns. (5a,b) may be written as

$$T_{\mu\nu}(\nu, t) = \frac{1}{\pi}\int \frac{t_{\mu\nu}(\nu', t)}{\nu' - \nu - i\epsilon}\, d\nu' \qquad (2.7a)$$

$$W_\mu(\nu, t) = \frac{1}{\pi}\int \frac{\omega_\mu(\nu', t)}{\nu' - \nu - i\epsilon}\, d\nu' \qquad (2.7b)$$

so that from Eqns. (6,6') we get

$$A(\nu, t) = \frac{1}{\pi}\int \frac{a(\nu', t)}{\nu' - \nu - i\epsilon}\, d\nu' \qquad (2.8)$$

$$L(\nu, t) = \frac{1}{\pi}\int \frac{l(\nu', t)}{\nu' - \nu - i\epsilon}\, d\nu' \qquad (2.9)$$

The equal time commutation relation of the currents consistent with the assumptions of covariance of $T_{\mu\nu}$ and W_μ is

$$[j_0^\alpha(x), j_\nu^\beta(0)]_{x_0=0} = C^{\alpha\beta\gamma}\, j_\nu^\gamma(x)\, \delta^3(\underline{x}) . \qquad (2.10)$$

It can readily be seen that the presence of Schwinger( ) terms in the above equal time commutator, would necessarily give rise to non covariant terms in the first integral on the r.h.s. of Eqn. (3), in contradiction with the above mentioned assumptions of covariance.

We can write

$$\langle p'| j_\mu^\alpha(0)|p\rangle = F_1^\alpha(t)\, P_\mu + F_2^\alpha(t)\, \Delta_\mu \tag{2.11}$$

where $F_2^\alpha(t) = 0$ if $\partial^\mu j_\mu^\alpha = 0$.

From Eqns. (3,6,6',10,11) on comparison of the coefficient of next line

P_μ , we get respectively

$$\nu A = C^{\alpha\beta\gamma} F_1^\gamma(t) - L \tag{2.12}$$

$$\nu a = - l \tag{2.13}$$

From Eqns. (9,13) we get

$$L(\nu,t) = -\frac{1}{\pi}\int \frac{\nu a(\nu',t)}{\nu'-\nu}\, d\nu' \tag{2.14}$$

Substituting into Eqn. (12), the expressions for $A(\nu,t)$ and $L(\nu,t)$ given by Eqns. (8) and (9) respectively, and rearranging we finally derive

$$\frac{1}{\pi}\int a(\nu',t)\, d\nu' = C^{\alpha\beta\gamma} F^\gamma(t) \tag{2.15}$$

To derive the superconvergence relation (1.3) from the above equation, we shall use the hypothesis of pole dominance of currents as expressed by Eqn. (1.2).

Defining

$$\langle 0| j^\alpha(0)| \rho_{i(\alpha)}\rangle = f_{i(\alpha)} . \tag{2.16}$$

we can write Eqn. (15) as

$$\frac{1}{\pi}\sum_{i(\alpha)\,j(\beta)} \frac{1}{q^2-m^2_{i(\alpha)}}\,\frac{1}{k^2-m^2_{j(\beta)}}\; f_{i(\alpha)}\, f_{j(\beta)} \int a^{i(\alpha)\,j(\beta)}(\nu',t)\,d\nu' \approx C^{\alpha\beta\gamma}F^{\gamma}(t) \tag{2.17}$$

for q^2, k^2 near $m^2_{i(\alpha)}$ $m^2_{j(\beta)}$ respectively.

$a^{i(\alpha),j(\beta)}$ is the absorptive part of the appropriate amplitude $A^{i(\alpha),j(\beta)}$ associated with the physical process next line $j(\beta)+p \rightarrow i(\alpha)+p'$. As the r.h.s. of the above equation is a function of t only, and does not contain any singularities in q^2 and k^2, we must have

$$\int a^{i(\alpha)\,j(\beta)}(\nu',t)\,d\nu' = 0 \tag{2.18}$$

It is also readily seen from Eqn. (17) that the pole dominance of currents is only approximately true if we have next line $F^{\alpha}(t) \neq 0$. From a slightly different viewpoint this can be rendered plausible as follows:

$$F^{\alpha}(t) \neq 0 \Rightarrow \langle bp' |\, j^{\alpha}(0)\, | ap \rangle \neq 0$$

From the requirement of crossing symmetry for the local operator $j^{\alpha}(x)$

$$\langle bp' |\, j^{\alpha}(0)\, | ap' \rangle \neq 0 \Rightarrow \langle 0 |\, j^{\alpha}(0)\, | a,b \rangle \neq 0$$

Thus we see that the Eqn. (1.2) can at best be only approximately true.

3. Derivation of the Superconvergence Relations from 'Analyticity' and Unitarity

It has been suggested in ref. (39), that sum rules of the general type given by Eqn. (2.18), for the amplitudes describing the scattering of strongly interacting particles, may alternatively be derived from the requirements of analyticity of the amplitudes, and appropriate asymptotic behaviour of the amplitudes deducible from unitarity.

To illustrate the above statement, we consider an amplitude $A(\nu,t)$ which is analytic in the complex ν plane except for singularities on the real axis, with the following asymptotic behaviour for large ν

$$|A(\nu,t)| < K\,\nu^{Re\,\alpha(t)} \quad : \quad Re\,\alpha(t) < 0$$

Then we can write the unsubtracted dispersion relation

$$Re\,A(\nu,t) = \frac{1}{\pi} P \int \frac{Im\,A(\nu',t)}{\nu'-\nu}\,d\nu'$$

If we further assume that $Re\,\alpha < -1$ so that

$\lim\limits_{\nu\to\infty} \nu|A(\nu,t)| = 0$ i.e. $A(\nu,t)$ is superconvergent,

we get from Eqn. (3.0) the relation

$$\int Im\,A(\nu,t)\,d\nu = 0 \tag{3.1}$$

From the point of view of Regge pole theory, we would require the t- channel to be dominated by a trajectory with next line $Re\,\alpha(t) < -1$ in order that the superconvergence relation (1) may hold for the amplitude $A(\nu,t)$, describing the scattering of particles with zero spin. However, the situation is radically different for the case in which particles with higher spin are present. In that case, as will be seen in the following, additional convergent factors are present because of the higher spins which enable us to write superconvergence relations for certain amplitudes under less restrictive assumptions about $Re\,\alpha(t)$.

The example of $\rho\pi$ scattering (examined in detail elsewhere[41]), serves rather well as an illustration of the above mentioned points.

We shall describe it briefly in the following.

The T matrix element for the process is expressible in terms of <u>four</u> invariant amplitudes, taking due account of time reversal invariance, i.e.

$$T = \epsilon_1 . P\, \epsilon_2 . P\, A + (\epsilon_1 . P\, \epsilon_2 . Q + \epsilon_1 . Q\, \epsilon_2 . P) B + \epsilon_1 . Q\, \epsilon_2 . Q\, C + \epsilon_1 . \epsilon_2\, D \tag{3.2}$$

where $\epsilon_1, q\,(p')$ are the polarisation and momentum respectively of the outgoing ρ meson (pion), and $\epsilon_2, k\,(p)$ are the corresponding specifications for the incoming particles. P, Q are as defined in section (2), above. The invariant amplitudes A, B, C, D are functions of $\nu\ (= P.Q)$ and $t = (p'-p)^2$.

We shall require for the following argument the 'orthogonal' decomposition of the invariant matrix T which is

$$T = \alpha I_\alpha + \beta I_\beta + \gamma I_\gamma + \delta I_\delta$$

where

$$\begin{aligned} I_\alpha &= (\epsilon_1 . P')(\epsilon_2 . P') \\ I_\beta &= (\epsilon_1 . P')(\epsilon_2 . Q) + (\epsilon_2 . P')(\epsilon_1 . Q) \\ I_\gamma &= (\epsilon_1 . Q)(\epsilon_2 . Q) \\ I_\delta &= (\epsilon_1 . N)(\epsilon_2 . N) \end{aligned} \tag{3.3}$$

$$P' = P - \frac{\nu}{Q^2} Q \qquad N_\mu = \epsilon_{\mu\nu\lambda\sigma} P^\nu Q^\lambda \Delta^\sigma$$

The amplitudes A, B, C, D are expressible in terms of the amplitudes $\alpha, \beta, \gamma, \delta$ as follows:

$$A = \alpha + \frac{1}{4} Q^2 \Delta^2 \beta$$
$$B = -\frac{4\nu}{Q^2}\alpha + \beta - \nu \Delta^2 \delta$$
$$C = \left(\frac{2\nu}{Q^2}\right)^2 \alpha - \frac{2\nu}{Q^2}\beta + \gamma + \left[\frac{P'^2}{4}(\Delta^2 - Q^2) + \frac{4\nu^2 \Delta^2}{Q^2}\right]\delta \tag{3.4}$$
$$D = -\frac{1}{4} P'^2 Q^2 \Delta^2 \delta .$$

We now require that the total cross section be greater than the individual contributions of the four amplitudes $\alpha, \beta, \gamma, \delta$. The total cross section is the sum of the individual contributions, and because of the particular choice of invariants there are no interference terms.

Thus we have for large s

$$s^4 \int |\alpha|^2 dt < K s^2 \sigma_T , \quad s^2 \int Q^2 |\beta|^2 dt < K s^2 \sigma_T$$
$$\int Q^2 |\gamma|^2 t^2 dt < K s^2 \sigma_T , \quad s^4 \int |\delta|^2 t^4 dt < K s^2 \sigma_T \tag{3.5}$$

Assuming the diffraction peak has a constant shape(*) (independent of s), and expressing the results in terms of A, B, C, D we get the following asymptotic behaviour for large ν and fixed t $(t>0)$.

$$\begin{aligned} |A| &< C_1 s^{-1} \\ |B| &< C_1 \\ |C| &< C_1 s \\ |D| &< C_1 s \end{aligned} \tag{3.6}$$

The above given upper limit on $|A|$ does not directly enable us to write down the superconvergence relation for A. However, if we adopt the point of view of Regge Pole theory,

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* This is not in contradiction with the Regge pole theory ideas introduced in the following, according to which there is only logarithmic shrinking of the diffraction peak.

and consider the parts of the amplitudes corresponding to $I=1$ in the t channel denoted in the following by $A^{(1)}, B^{(1)}, \ldots$ etc., then since for large ν the amplitudes $A^{(1)}, \ldots$ are dominated by the ρ meson trajectory rather than the corresponding Pomeranchuk( ) trajectory for the amplitudes $A^{(1)}, B^{(1)}, \ldots$ etc., we obtain an additional factor of $\nu^{Re\,\alpha_\rho - Re\,\alpha_P}$ in the limit $\nu \to \infty$ for the amplitudes $A^{(1)} \ldots$ etc.

Thus we have

$$\begin{aligned}
|A^{(1)}| &< C_1\, \nu^{Re\,\alpha_\rho - Re\,\alpha_P - 1} \\
|B^{(1)}| &< C_1\, \nu^{Re\,\alpha_\rho - Re\,\alpha_P} \\
|C^{(1)}| &< C_1\, \nu^{Re\,\alpha_\rho - Re\,\alpha_P + 1} & \qquad Re\,\alpha_P(0) = 1 \\
|D^{(1)}| &< C_1\, \nu^{Re\,\alpha_\rho - Re\,\alpha_P + 1} & \qquad Re\,\alpha_\rho(0) < 1
\end{aligned} \tag{3.7}$$

so that $\nu |A^{(1)}(\nu,t)| \to 0$ as $\nu \to \infty$ for $t \in I$

if we assume that $Re\,\alpha_\rho(t) - Re\,\alpha_P(t) < 0$ for ~~all~~ $t \in I$.

We can now write the superconvergence relation

$$\int_{-\infty}^{+\infty} Im\, A^{(1)}(\nu,t)\, d\nu = 0$$

From crossing symmetry,

$$Im\, A^{(0,2)}(\nu,t) = -\, Im\, A^{(0,2)}(-\nu,t)$$

so that

$$\int_{-\infty}^{+\infty} Im\, A^{(0,2)}(\nu,t)\, d\nu = 0$$

Thus we have derived the relation

$$\int Im\, A(\nu,t)\, d\nu = 0 \tag{3.8}$$

Approximating the l.h.s. of Eqn. (8) by the contributions of π and ω, we get

$$4 g^2_{\rho\pi\pi} - m_\rho^2\, g^2_{\omega\rho\pi} = 0 \tag{3.8'}$$

where the couplings $g_{\rho\pi\pi}$, $g_{\omega\rho\pi}$ are defined as follows:

$$\begin{aligned} \rho\pi\pi : &\quad g_{\rho\pi\pi}\, \epsilon_{ijk}\, \rho^i_\mu\, \pi^j\, \partial^\mu \pi^k \\ \omega\rho\pi : &\quad g_{\omega\rho\pi}\, \epsilon_{\mu\nu\lambda\sigma}\, \partial^\mu \omega^\nu\, \partial^\lambda \underline{\rho}^\sigma \cdot \underline{\pi} \end{aligned} \tag{3.9}$$

From the width of ρ meson $(\sim 150\ \text{Mev})$ we have $g^2_{\rho\pi\pi} \sim 12\pi$. Calculating the partial width $\Gamma(\omega \rightarrow 3\pi)$ from the Gell-Mann, Sharp and Wagner model[42] using the value of $g_{\omega\rho\pi}$ obtained from Eqn. (8'), we have

$$\Gamma_{G.S.W.}(\omega \rightarrow 3\pi) \sim 8.2\ \text{Mev}$$

On the other hand

$$\Gamma_{exp.}(\omega \rightarrow 3\pi) \sim 10 \pm 2\ \text{Mev}$$

If we now further assume that the most dominant $I=2$ Regge trajectory $\alpha^{(2)}(t)$ is such that

$$Re\, \alpha^{(2)}(t) < 0$$

then we may write for t near zero

$$\int_0^\infty Im\, B^{(2)}(\nu, t)\, d\nu = 0 \tag{3.10}$$

$$\int_0^\infty \nu\, Im\, A^{(2)}(\nu, t)\, d\nu = 0 \tag{3.11}$$

Eqns. (8',10,11) for $t=0$ can no longer be saturated with ω, ϕ, π in a non trivial manner. Indeed we get $g_{\omega\rho\pi} = g_{\phi\rho\pi} = g_{\rho\pi\pi} = 0$.

Frampton and Taylor[43] have examined the saturation of Eqns. (8',10,11) at $t=0$ and their first derivatives at $t=0$ by considering the contributions of π, ω, A_1, A_2 states.

Taking the $A_1\rho\pi$ and $A_2\rho\pi$ couplings as

$$\begin{aligned} A_1\rho\pi : &\quad g_{A_1\rho\pi} \left\{ \epsilon^\rho \cdot \epsilon^A + \frac{C}{m_\rho m_A}\, \epsilon^\rho \cdot p^{A_1}\, \epsilon^{A_1} \cdot p^\rho \right\} \delta_{ij} \\ A_2\rho\pi : &\quad g_{A_2\rho\pi}\, \epsilon^\rho_\mu\, \epsilon^{A_2}_{\nu\lambda}\, p^\lambda_{(\rho)}\, p^{(\rho)}_\sigma\, p^{A_2}_\tau\, \epsilon^{\mu\nu\sigma\tau}\, i\epsilon_{ijk} \end{aligned} \tag{3.12}$$

where $|g_{A_1\rho\pi}| \sim 1.9$ Bev , $C \sim 10$ and $|g_{A_2\rho\pi}| \sim 20\ \mathrm{Gev}^{-2}$

which are consistent with the widths of A_1 and A_2

$$\Gamma(A_1 \to \rho\pi) = \frac{p_\pi^{(1)}}{12\pi m_{A_1}^2}\, g_{A_1\rho\pi}^2 \left(3 + 0.2\,C + .01\,C^2\right) \sim 130\ \mathrm{Mev.}$$

$$\Gamma(A_2 \to \rho\pi) = \frac{p_\pi^{(2)}}{20\pi m_{A_2}^2}\, g_{A_2\rho\pi}^2 \frac{1}{16 m_{A_2}^2}\left[\left(m_\rho^2 + m_{A_2}^2 - m_\pi^2\right)^2 - 4 m_\rho^2 m_{A_2}^2\right] \sim 80\ \mathrm{Me}$$

the above mentioned authors find a satisfactory saturation of the equations at $t=0$ and the derivative of Eqn. (8') at $t=0$. The saturation of the derivatives of Eqns. (10,11) at $t=0$ are not so satisfactory however, and probably requires further higher spin contributions.

4. Superconvergence Relations for Pion Photoproduction Off Nucleons

α . Preliminary discussion

In this section we shall derive superconvergence relations for pion photoproduction off nucleons, by determining the asymptotic behaviour in s (for fixed u) of the appropriate amplitudes, through a study of the possible Regge trajectories arising in the u- channel.

We define

$$s = (p+k)^2$$
$$u = (k-p')^2$$
$$t = (p-p')^2$$

so that $\quad s+t+u = -2m^2 - m_\pi^2 \quad .$

The amplitude for photoproduction can be written as [43,44] $\epsilon^\mu T_\mu$ where

$$T_\mu = \sum_{i=1}^{4} A^i M^i_\mu$$
$$M^1 = i\gamma_5 [P_\mu - \not{Q}\gamma_\mu + Q_\mu]$$
$$M^2 = 2i\gamma_5 [2m\nu_1 P_\mu + m\nu (\tfrac{1}{2} Q_\mu + \tfrac{3}{4}\Delta_\mu)]$$
$$M^3 = \gamma_5 [2m\nu_1 \gamma_\mu - (\not{Q} - im)(Q_\mu + \tfrac{1}{2}\Delta_\mu)] \tag{4.1}$$
$$M^4 = 2\gamma_5 [-m\nu\gamma_\mu - (\not{Q} - im)P_\mu] - 2m M^1$$
$$P = \tfrac{1}{2}(p+p') \quad , \quad Q = \tfrac{1}{2}(q+k) \quad , \quad \Delta = p - p'$$
$$m\nu = -P\cdot Q \quad , \quad 2m\nu_1 = q\cdot k$$

We have omitted writing the nucleon spinor wave functions for the sake of simplicity.

The gauge invariant amplitudes A^i thus constructed are devoid of kinematic singularities[45]. This can be seen by verifying that all possible vectors, constructed from the set of available vectors (excluding ϵ_μ) when contracted with the photon polarisation vector ϵ_μ are expressible in terms of $\epsilon . M^i$ with non singular coefficients.

• The Two Component Spinor Formalism and the Appropriate Amplitudes.

We shall now introduce the two component spinor formalism and the relevant amplitudes suitable for the purpose of the angular momentum decomposition in the u- channel.

The spinor part of the nucleon (antinucleon) wave function satisfies the equation

$$(i\not{p}+m)\,u(p) = 0 \qquad : \not{p} = p^{\mu}\gamma_{\mu}$$

where the upper and lower signs refer to the nucleon and anti-nucleon respectively (or vice versa).

In the Dirac-Pauli representation, which has the important feature that the antinucleon components of the nucleon spinor wave function are negligible in the non relativistic limit, we have

$$\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \underline{\gamma} = \begin{pmatrix} 0 & -i\underline{\sigma} \\ i\underline{\sigma} & 0 \end{pmatrix}$$

The invariant normalisation of the wave function $u^r(p)$, is given by

$$\bar{u}^r(p)\,u^s(p) = \pm 2m\,\delta_{rs} \tag{4.2}$$

Writing the spinor $u^r(p)$ in terms of the 'large' and small components as

$$u^r(p) = \begin{pmatrix} u^r_a(\underline{p}) \\ u^r_b(\underline{p}) \end{pmatrix}$$

the Dirac equation becomes

$$\underline{\sigma}.\underline{p}\;u^r_b(\underline{p}) + m\,u^r_a(\underline{p}) = E_p\,u^r_a(\underline{p})$$

$$\underline{\sigma}.\underline{p}\;u^r_a(\underline{p}) - m\,u^r_b(\underline{p}) = E_p\,u^r_b(\underline{p})$$

The second equation allows us to express the small component in terms of the large component

$$u_b^r(\underline{p}) = \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\, u_a^r(\underline{p}) \qquad (4.3)$$

From equation (2) we get

$$u_a^{r*}(\underline{p})\, u_a^s(\underline{p}) = (E+m)\,\delta_{rs} \qquad (4.4)$$

The polarisation vector of the photon ϵ_μ will be taken to have its fourth component vanishing, viz. $\epsilon_\mu = (0, \underline{\epsilon})$, so that the condition of transversality of the photon polarisation is $\underline{k}\cdot\underline{\epsilon} = 0$.

The invariant matrix $\epsilon^\mu T_\mu$ can now be written in the Pauli spinor formalism as follows[46]

$$\begin{aligned}\epsilon^\mu T_\mu &= \sum_i \epsilon^\mu A^i M_\mu^i \\ &= \bar{u}_a^r(\underline{q})\Big(i\underline{\sigma}\cdot\underline{\epsilon}\,\mathcal{F}_1 + \frac{\underline{\sigma}\cdot\underline{q}\;\underline{\sigma}\cdot(\underline{k}\wedge\underline{\epsilon})}{|\underline{q}|\,|\underline{k}|}\,\mathcal{F}_2 + \frac{i\,\underline{\sigma}\cdot\underline{k}\;\underline{q}\cdot\underline{\epsilon}}{|\underline{q}|\,|\underline{k}|}\,\mathcal{F}_3 \\ &\quad + \frac{i\,\underline{\sigma}\cdot\underline{q}\;\underline{q}\cdot\underline{\epsilon}}{|\underline{q}|^2}\,\mathcal{F}_4\Big)\, u_a^s(\underline{k})\end{aligned} \qquad (4.5)$$

where $\underline{q}$, $\underline{k}$ are the momenta of the pion and the photon respectively, in the centre of mass system.

$$|\underline{q}| = \left\{[u+(m+m_\pi)^2][u-(m-m_\pi)^2]/-4u\right\}^{1/2}$$

$$|\underline{k}| = -\frac{u+m^2}{2\sqrt{-u}}$$

The following relations expressing $\mathcal{F}_i$'s as linear combinations of A_i's can be derived in a straightforward manner[47].

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$$\mathcal{F}_1 = \frac{W-m}{8\pi W}\left[(E_1+m)(E_2+m)\right]^{\frac{1}{2}}\left[A_1 + (W-m)A_4 + \frac{t+m_\pi^2}{2(W-m)}(A_3-A_4)\right]$$

$$\mathcal{F}_2 = \frac{W-m}{8\pi W}\left(\frac{E_1+m}{E_2+m}\right)^{\frac{1}{2}}|\underline{q}|\left[-A_1 + (W+m)A_4 + \frac{t+m_\pi^2}{2(W+m)}(A_3-A_4)\right]$$

$$\mathcal{F}_3 = \frac{W-m}{8\pi W}\left[(E_1+m)(E_2+m)\right]^{\frac{1}{2}}|\underline{q}|\left[(W-m)A_2 + A_3 - A_4\right]$$

$$\mathcal{F}_4 = \frac{W-m}{8\pi W}\left(\frac{E_1+m}{E_2+m}\right)^{\frac{1}{2}}|\underline{q}|^2\left[-(W+m)A_2 + A_3 - A_4\right] \qquad (4.6)$$

where $W = \sqrt{-u}$, $E_1 = (|\underline{k}|^2 + m^2)^{\frac{1}{2}}$, $E_2 = (|\underline{q}|^2 + m^2)^{\frac{1}{2}}$

γ. Multipole Expansions for the Transition Amplitudes $\mathcal{F}_i$.

For a particle of spin 1 , the three states of 'linear' polarisation are denoted by $|i\rangle$ $(i=1,2,3)$. The spin operator S_i is defined as

$$\langle j | S_i | k\rangle = i\,\epsilon_{ikj}$$

The states of circular polarisation are

$$|\pm 1\rangle = \pm\frac{1}{\sqrt{2}}\left(|1\rangle \pm i\,|2\rangle\right) \quad , \quad |0\rangle = |3\rangle$$

They have the following properties

$$S_3|\pm 1\rangle = \pm|\pm 1\rangle \;, \quad S_3|0\rangle = 0 \quad , \quad S_\pm|\pm 1\rangle = 0$$

$$S_\pm|\mp 1\rangle = -\sqrt{2}\,|0\rangle \,, \quad S_\pm|0\rangle = -\sqrt{2}\,|\pm 1\rangle$$

For a photon with momentum along the 3- axis, only the states $|\pm 1\rangle$ exist as is required by the radiation guage condition $\underline{k}.\underline{\epsilon} = 0$. The eigenstates of $\underline{\ell}^2, \ell_3, S_3$ (where $\underline{\ell}$ is the orbital angular momentum), will be represented as $|\ell, \ell_3, S_3\rangle$. The corresponding eigenvalues for the state are $l(l+1),\ l_3,\ s_3$. It is possible to transform to the unitary equivalent representation for which the basic states are eigenstates of $\underline{j}^2,\ j_3$ and $\underline{l}^2$, where $\underline{j} = \underline{l} + \underline{s}$.

The projection operators corresponding to the three possible eigenvalues of $\underline{j}^2$ for a given eigenvalue $l(l+1)$ of $\underline{l}^2$ are

$$\Lambda_{j=l+1} = \frac{(\underline{l}\cdot\underline{s} + l + 1)(\underline{l}\cdot\underline{s}+1)}{(2l+1)(l+1)}$$
$$\Lambda_{j=l} = \frac{(l - \underline{l}\cdot\underline{s})(\underline{l}\cdot\underline{s} + l + 1)}{l(l+1)} \tag{4.7}$$
$$\Lambda_{j=l-1} = \frac{(\underline{l}\cdot\underline{s} - 1)(\underline{l}\cdot\underline{s}+1)}{l(2l+1)}$$

Using the projection operators given above, we can now readily express the states $|j, j_3, l\rangle$ as linear combinations of the states $|l, l_3, s_3\rangle$.

$$|j,j_3,j-1\rangle = \left[\frac{(j+j_3-1)(j+j_3)}{(2j-1)2j}\right]^{\frac{1}{2}} |j-1, j_3-1, 1\rangle + \left[\frac{(j-j_3)(j+j_3)}{(2j-1)j}\right]^{\frac{1}{2}} |j-1, j_3, 0\rangle + \left[\frac{(j-j_3-1)(j-j_3)}{(2j-1)2j}\right]^{\frac{1}{2}} |j-1, j_3+1, -1\rangle \tag{4.8}$$

$$|j,j_3,j\rangle = -\left[\frac{(j+j_3)(j-j_3+1)}{2j(j+1)}\right]^{\frac{1}{2}} |j, j_3-1, 1\rangle + \frac{j_3}{[j(j+1)]^{\frac{1}{2}}} |j, j_3, 0\rangle + \left[\frac{(j-j_3)(j+j_3+1)}{2j(j+1)}\right]^{\frac{1}{2}} |j, j_3+\cdots$$

$$|j,j_3,j+1\rangle = \left[\frac{(j-j_3+1)(j-j_3+2)}{(2j+2)(2j+3)}\right]^{\frac{1}{2}} |j, j_3-1, 1\rangle - \left[\frac{(j-j_3+1)(j+j_3+1)}{(j+1)(2j+3)}\right]^{\frac{1}{2}} |j, j_3, 0\rangle + \left[\frac{(j+j_3+2)(j+j_3+\cdots}{(2j+2)(2j+3)}\right. |j, j_3+1, -1\rangle$$

The state containing a photon of momentum $\underline{k}$ and polarisation $\underline{\epsilon}$ may be written as

$$|\underline{k},\underline{\epsilon}\rangle = C \sum_{l,l_3} \sum_{\nu=0,\pm1} Y_l^{l_3}(\underline{k})\, \epsilon_\nu\, |l\ l_3\ \nu\rangle \ . \tag{4.9}$$

We shall adopt the normalisation of Lippman and Schwinger according to which

$$\frac{1}{(2\pi)^3}\int \frac{d^3\underline{p}}{2E_p\, 2|\underline{p}|}\;\delta(E_p+|\underline{p}|-W)\,|\underline{p},\epsilon\rangle\langle\underline{p},\epsilon| = 1_\epsilon \tag{4.10}$$

where 1_ϵ denotes the identity operator in the subspace of states with polarisation $\underline{\epsilon}$.

With this normalisation convention we must take

$$|C|^2 = 4\cdot(2\pi)^3\,\frac{W}{|\underline{k}|} \tag{4.11}$$

if we are to require $\langle l\, l_3\, \nu\,|\,l'\, l_3'\, \nu'\rangle = \delta_{ll'}\,\delta_{l_3 l_3'}\,\delta_{\nu\nu'}$

From equations (8,9) we obtain

$$|\underline{k},\underline{\epsilon}\rangle = C\sum_{j\,j_3}\sum_{\nu=\pm1}\epsilon_\nu\Big\{ Y_{j-1}^{j_3-\nu\,*}(\underline{k})\left(\frac{(j+\nu j_3-1)(j+\nu j_3)}{2j(2j-1)}\right)^{1/2}|j,j_3,j-1\rangle$$

$$+\,Y_{j+1}^{j_3-\nu\,*}(\underline{k})\left(\frac{(j-\nu j_3+2)(j-\nu j_3+1)}{2(j+1)(2j+3)}\right)^{1/2}|j,j_3,j+1\rangle$$

$$+\,\nu\, Y_{j}^{j_3-\nu\,*}(\underline{k})\left(\frac{(j-\nu j_3+1)(j+\nu j_3)}{2j(j+1)}\right)^{1/2}|j,j_3,j\rangle \tag{4.12}$$

$+$ term involving ϵ_3 .

Assuming $\underline{k}$ to be in the direction of the $3-$ axis, we have $\epsilon_3 = 0$, $Y_l^m(\theta=0) = \left(\frac{2l+1}{4\pi}\right)^{1/2}\delta_{m0}$ and

$$|\underline{k},\underline{\epsilon}\rangle = C\sum_{j,j_3} A_{jj_3\lambda}\sqrt{2j+1}\;|j\,j_3\,\lambda\rangle$$

where $A_{jj_3\lambda} = \frac{1}{\sqrt{8\pi}}\left(\delta_{j_3,+1}\,\epsilon_+ + \delta_{j_3,-1}\,\epsilon_-\right)\left(\delta_{\lambda,(-1)^j} + \nu\,\delta_{\lambda,(-1)^{j+1}}\right)$

λ designates the parity of the state $|j,j_3,\lambda\rangle$ which is defined as follows:

$$|j\, j_3\, (-1)^{j+1}\rangle = |j\, j_3\, j\rangle$$

$$|j\, j_3\, (-1)^{j}\rangle = \frac{1}{\sqrt{2j+1}}\left(\sqrt{j+1}\,|j, j_3, j-1\rangle + \sqrt{j}\,|j, j_3, j+1\rangle\right) \qquad (4.13)$$

~~and~~ .

The states $|j\, j_3\, \lambda\rangle$ are normalised eigenstates of $\underline{j}^2\ j_3$ and the parity operator. The above analysis is the quantum mechanical analogue of the resolution of a plane electromagnetic wave into the magnetic $(\lambda = (-1)^{j+1})$ and the electric $(\lambda = (-1)^{j})$ multipoles in classical electromagnetic theory.

Now we combine the total angular momentum of the photon and the nucleon spin to get

$$\underline{J} = \underline{j} + \tfrac{1}{2}\underline{\sigma}$$

The state $|\underline{k}, \underline{\epsilon}\rangle = |\underline{k} = -\underline{p}, \underline{\epsilon}, \sigma_3\rangle$ can thus be written as a linear combination of the eigenstates of $\underline{J}^2,\ J_3$ and the parity operator, viz.

$$\begin{aligned}|\underline{k}, \underline{\epsilon}\rangle = C \sum_{j j_3 \lambda} A_{j j_3 \lambda} \Big(&\delta_{\sigma_3, 1} \sqrt{j - j_3}\, |j - \tfrac{1}{2}, j_3 + \tfrac{1}{2}, \lambda\rangle \\ &+ \delta_{\sigma_3, -1} \sqrt{j + j_3}\, |j - \tfrac{1}{2}, j_3 - \tfrac{1}{2}, \lambda\rangle + \delta_{\sigma_3, 1} \sqrt{j + j_3 + 1}\, |j + \tfrac{1}{2}, j_3 + \tfrac{1}{2}, \lambda\rangle \\ &- \delta_{\sigma_3, -1} \sqrt{j - j_3 + 1}\, |j + \tfrac{1}{2}, j_3 - \tfrac{1}{2}, \lambda\rangle \Big)\end{aligned} \qquad (4.14)$$

In a similar way the final state $|\underline{q}, -\underline{p}', \sigma_3'\rangle$ may be written as linear combination of the eigenstates of $\underline{J}^2, J_3, \underline{\ell}^2$

$$\langle \underline{q}=-\underline{p}',\sigma_3'| = C'^{*} \sum_{l,l_3} \frac{1}{\sqrt{2l+1}} Y_l^{l_3}(\underline{q}) \Big(\delta_{\sigma_3',+1} \sqrt{l-l_3} \langle l-\tfrac{1}{2}, l_3+\tfrac{1}{2}, l| + \delta_{\sigma_3',-1} \sqrt{l+l_3} \langle l-\tfrac{1}{2}, l_3-\tfrac{1}{2}, l| + \delta_{\sigma_3',1} \sqrt{l+l_3+1} \langle l+\tfrac{1}{2}, l_3+\tfrac{1}{2}, l| - \delta_{\sigma_3',-1} \sqrt{l-l_3+1} \langle l+\tfrac{1}{2}, l_3-\tfrac{1}{2}, l| \tag{4.15}$$

where

$$|C'|^2 = 4\,(2\pi)^3 \frac{W}{|\underline{q}|}$$

The transition operator in the spin space of the nucleons is

$$\mathcal{T} = \sum_{\sigma_3 \sigma_3'} |\sigma_3'\rangle \langle \underline{p}'=-\underline{q}, \sigma_3'| T | \underline{p}=-\underline{k}, \sigma_3\rangle \langle \sigma_3| \tag{4.16}$$

From equations (14,15,16), making use of the following relations

$$|\pm 1\rangle\langle \pm 1| = \tfrac{1}{2}(1 \pm \sigma_3) \qquad |\pm 1\rangle\langle \mp 1| = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)$$

$$\sum_{l_3=-l}^{l} Y_l^{l_3}(\underline{p}')\, Y_l^{l_3 *}(\underline{p}) = \frac{2l+1}{4\pi} P_l(\cos\theta) \qquad \cos\theta = \frac{\underline{q}\cdot\underline{k}}{|\underline{q}|\,|\underline{k}|}$$

$$\sum_{J_3} \frac{J \pm J_3 + 1}{2(J+1)} Y_{J+\frac{1}{2}}^{J_3 \pm \frac{1}{2}}(\underline{p}')\, Y_{J+\frac{1}{2}}^{J_3 \pm \frac{1}{2} *}(\underline{p}) = \frac{1}{8\pi}\Big(2J+1 \mp 2i\frac{\partial}{\partial \varphi_3}\Big) P_{J+\frac{1}{2}}(\cos\theta)$$

(where φ_3 is the angle between the projection of $\underline{q}$ on the ($1,2$) plane and a fixed line in the plane. φ_1, φ_2 are defined analogously).

$$\sum_{J_3} \frac{[(J+J_3+1)(J-J_3+1)]^{\frac{1}{2}}}{2(J+1)} Y_{J+\frac{1}{2}}^{J_3 \pm \frac{1}{2}}(\underline{p}')\, Y_{J+\frac{1}{2}}^{J_3 \mp \frac{1}{2} *}(\underline{p}) = \frac{i}{4\pi}\Big(\frac{\partial}{\partial\varphi_1} \pm i\frac{\partial}{\partial\varphi_2}\Big) P_{J+\frac{1}{2}}(\cos\theta)$$

$$-i\underline{\sigma}\cdot\frac{\partial}{\partial\underline{\varphi}} P_{J+\frac{1}{2}}(\cos\theta) = i\,\frac{\underline{\sigma}\cdot\underline{k}\wedge\underline{q}}{|\underline{k}|\,|\underline{q}|} P'_{J+\frac{1}{2}}(\cos\theta)$$

$$\underline{\sigma}\cdot(\underline{q}\wedge\underline{k}) = i[\underline{q}\cdot\underline{k} - (\underline{\sigma}\cdot\underline{q})(\underline{\sigma}\cdot\underline{k})]$$

$$\ell P_\ell(x) = x P'_\ell(x) - P'_{\ell-1}(x)$$

$$(\ell+1) P_\ell(x) = P'_{\ell+1}(x) - x P'_\ell(x)$$

we finally obtain

$$\mathcal{J} = i\frac{c'^* c}{4\pi}\sum\frac{1}{\ell(\ell+1)}\Big\{T_\ell^{--}\Big((\ell+1)P'_{\ell-1}I_1 + \ell P'_\ell I_2 + P''_{\ell-1}I_3 - P''_\ell I_4\Big) + T_\ell^{++}\Big(\ell P'_{\ell+1}I_1 + (\ell+1)P'_\ell I_2 - P''_{\ell+1}I_3 + P''_\ell I_4\Big)\Big\}$$
$$+\frac{1}{\sqrt{\ell(\ell-1)}}T_\ell^{-+}\Big(P'_{\ell-1}I_1 + P''_{\ell-1}I_3 - P''_\ell I_4\Big) + \frac{1}{\sqrt{(\ell+1)(\ell+2)}}T_\ell^{+-}\Big(-P'_{\ell+1}I_1 - P''_{\ell+1}I_3 + P''_\ell I_4\Big)\Big\} \quad (4.17)$$

where

$$T_\ell^{\pm\pm} = \langle \ell\pm\tfrac{1}{2}, J_3, \ell / T / \ell\pm\tfrac{1}{2}, J_3, \ell\rangle$$

$$T_\ell^{\pm\mp} = \langle \ell\pm\tfrac{1}{2}, J_3, \ell / T / \ell\pm\tfrac{1}{2}, J_3, \ell\pm 1\rangle$$

Define $I_1 = i\underline{\sigma}\cdot\underline{\epsilon}$, $I_2 = \dfrac{\underline{\sigma}\cdot\underline{q}\ \underline{\sigma}\cdot\underline{k}\wedge\underline{\epsilon}}{|\underline{q}||\underline{k}|}$, $I_3 = \dfrac{i\underline{\sigma}\cdot\underline{k}\ \underline{q}\cdot\underline{\epsilon}}{|\underline{q}||\underline{k}|}$, $I_4 = \dfrac{i\underline{\sigma}\cdot\underline{q}\ \underline{q}\cdot\underline{\epsilon}}{|\underline{q}|^2}$

$$M_{\ell\pm} = -\frac{1}{\sqrt{|\underline{k}||\underline{q}|}}\frac{\pi}{\sqrt{\ell(\ell+1)}}T_\ell^{\pm\pm}.$$

$$E_{(\ell\pm 1)\mp} = \mp\frac{1}{\sqrt{|\underline{k}||\underline{q}|}}\frac{\pi}{\sqrt{\ell(\ell+1)}}T_{\ell\pm 1}^{\pm\mp}. \quad (4.18)$$

The multipole expansions for the amplitudes $\mathcal{F}_i$ are thus found to be

$$\mathcal{F}_1 = \sum_l l(M_{l+} + E_{l+})\, P'_{l+1}(x) + \left((l+1)M_{l-} + E_{l-}\right) P'_{l-1}(x) .$$

$$\mathcal{F}_2 = \sum_l \left((l+1)M_{l+} + l M_{l-}\right) P'_l(x)$$

$$\mathcal{F}_3 = \sum_l (E_{l+} - M_{l+})\, P''_{l+1}(x) + (E_{l-} + M_{l-})\, P''_{l-1}(x)$$

$$\mathcal{F}_4 = \sum_l \left(M_{l+} - E_{l+} - M_{l-} - E_{l-}\right) P''_l(x) . \tag{4.19}$$

where $x = \cos\theta_u = \dfrac{\underline{q}\cdot\underline{k}}{|\underline{q}||\underline{k}|} = \dfrac{-2ut + u^2 + u(2m^2 + m_\pi^2) + m^2(m^2 - m_\pi^2)}{(-u-m^2)\left[-u-(m-m_\pi)^2\right]^{1/2}\left[-u-(m+m_\pi)^2\right]^{1/2}} .$

5. Derivation of the Superconvergence relations.

The amplitudes $E_{l\pm}$, $M_{l\pm}$ introduced in the preceding section are functions of u only. They describe transitions produced by the electric and magnetic radiation respectively, resulting in final states of orbital angular momentum l and total angular momentum $l \pm \frac{1}{2}$. Thus the $\pm$ amplitudes receive contributions from the series of resonances with

$J^P = \frac{1}{2}^{\mp}, \frac{5}{2}^{\mp}, \ldots, \frac{3}{2}^{\pm}, \frac{7}{2}^{\pm}, \ldots$

Experimentally, the following trajectories are well established[48].

$I = \frac{1}{2}$ trajectories:

$\frac{1}{2}^+(940)$, $\frac{5}{2}^+(1688)$, ...

$\frac{1}{2}^+(1400)$, ...

$\frac{3}{2}^-(1525)$, $\frac{7}{2}^-(2190)$, ...

$\frac{1}{2}^-(1570)$, ...

$I=3/2$ trajectories:

$\frac{3}{2}^+(1236), \frac{7}{2}^+(1920), \ldots$

$\frac{1}{2}^-(1670), \ldots$

We consider the components of the amplitudes $\mathcal{F}_i$ corresponding to $I=3/2$ in the u- channel (denoted in the following as $\mathcal{F}_i^{(3/2)}$). Now, since the known resonances with $I=3/2$ can contribute only to the $(+)$ terms in the angular momentum decomposition of $\mathcal{F}_i$, we shall assume that <u>any</u> resonances of isospin $3/2$ with $J^P = \{\ell-\frac{1}{2}\}^{(-1)^{\ell+1}}$ are of sufficiently large mass to allow us to take for the corresponding trajectory functions $\alpha^{3/2(-)}(u)$:

$$-\frac{1}{2} > \max \operatorname{Re}\alpha^{3/2(-)}(u) \quad : \quad u \in I_1 \tag{5.1}$$

We shall also assume that for $u \in I_2$ the most dominant Regge trajectory with $I=3/2$ and $J^P = (\ell+\frac{1}{2})^{(-1)^\ell}$ is the N_δ (48). so that

$$\max \operatorname{Re}\alpha^{3/2(+)}(u) < \frac{1}{2} \quad : \quad u \in I_2 \tag{5.2}$$

Thus from the standard Regge theory the following high energy behaviour for the amplitudes $\mathcal{F}_i^{(3/2)}$ is obtained

$$\begin{aligned}
|\mathcal{F}_2^{(3/2)}(\nu,u)| &\to \nu^{\max\{\operatorname{Re}\alpha^{3/2(+)}(u),\ (\operatorname{Re}\alpha^{3/2(-)}(u)+1)\}}\,\nu^{-3/2} \\
|\mathcal{F}_3^{(3/2)}(\nu,u)| &\to \nu^{\max\{\operatorname{Re}\alpha^{3/2(+)}(u),\ (\operatorname{Re}\alpha^{3/2(-)}(u)+1)\}}\,\nu^{-3/2} \\
|\mathcal{F}_4^{(3/2)}(\nu,u)| &\to \nu^{\max\{\operatorname{Re}\alpha^{3/2(+)}(u),\ (\operatorname{Re}\alpha^{3/2(-)}(u)+1)\}}\,\nu^{-5/2}
\end{aligned} \tag{5.3}$$

We then obtain from (3)

$$\nu|\mathcal{F}_2^{(3/2)}| \to 0 \qquad \nu|\mathcal{F}_3^{(3/2)}| \to 0 \quad , \ \nu^2|\mathcal{F}_4^{(3/2)}| \to 0 \quad \text{as} \quad \nu \to \infty \ : u \in I_1 \cap I_2$$

The above asymptotic behaviour leads by virtue of the usual argument (section 3), to the following superconvergence relations

$$\int Im\, F_2^{(3/2)}(\nu, u)\, d\nu = 0$$
$$\int Im\, F_3^{(3/2)}(\nu, u)\, d\nu = 0$$
$$\int Im\, F_4^{(3/2)}(\nu, u)\, d\nu = 0$$
$$\int \nu\, Im\, F_4^{(3/2)}(\nu, u)\, d\nu = 0 \qquad : \; u \in I_1 \cap I_2 . \tag{5.4}$$

where F_i's are obtained from $\mathcal{F}_i'$s by dropping the factors singular at $u=0$. F_i's are thus given in terms of A_i's by the bracketted expressions on the r.h.s. of equations (19).

For the parts of the amplitudes $\mathcal{F}_i$ corresponding to $I=\frac{1}{2}$ in the u- channel, there is evidence of the presence of contributions from the $(+)$ as well as the $(-)$ terms in the angular momentum decomposition of $\mathcal{F}_i$, viz. the existence of $I=\frac{1}{2}$ resonances with $J^P = \{l+\frac{1}{2}\}^{(-1)^{l+1}}$ and $\{l-\frac{1}{2}\}^{(-1)^{l+1}}$. We now make the following assumptions suggested by the presently available data on nucleon resonances. The $(-)$ terms are dominated by the nucleon trajectory and the $(+)$ terms are dominated by the $N(1570)$ trajectory, so that

$$\frac{1}{2} > max\left\{ Re\, \alpha^{\frac{1}{2}(-)}(u),\ \left(Re\, \alpha^{\frac{1}{2}(+)}(u) + 1\right)\right\} \qquad : u \in I_1' \tag{5.5}$$

We thus derive

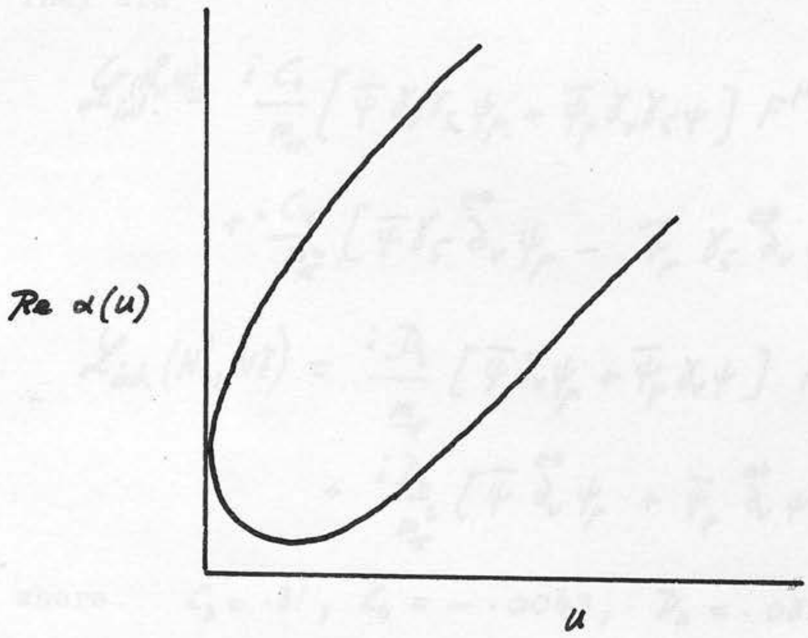
$$\begin{aligned} &\nu \left|\mathcal{F}_1^{(\frac{1}{2})}\right| \to 0 \\ &\nu^2 \left|\mathcal{F}_3^{(\frac{1}{2})}\right| \to 0 \\ &\nu \left|\mathcal{F}_4^{(\frac{1}{2})}\right| \to 0 \qquad \nu \to \infty \;,\quad u \in I_1' \end{aligned} \tag{5.6}$$

giving the superconvergence relations

$$\int Im\, F_1^{(\frac{1}{2})}(\nu, u)\, d\nu = 0$$
$$\int Im\, F_3^{(\frac{1}{2})}(\nu, u)\, d\nu = 0$$
$$\int \nu\, Im\, F_3^{(\frac{1}{2})}(\nu, u)\, d\nu = 0$$
$$\int Im\, F_4^{(\frac{1}{2})}(\nu, u)\, d\nu = 0 \qquad : u \in I_1' \tag{5.7}$$

6. Test of the Superconvergence Relations.

As a crude test of the validity of the superconvergence relations we shall restricit ourselves to the case $u=0$. It might be argued that we should not expect the relations (5.4) to hold at $u=0$, because for $u=0$ the condition (5.1) may no longer be valid. Indeed, according to the usual ideas a trajectory coincides at $u=0$ with the corresponding MacDowell symmetric trajectory(49) passing through resonances of opposite parity so that, for example, the condition (5.1) may not be valid at $u=0$. Thus it is quite possible that the interval $I_1 \cap I_2$ may not contain the point $u=0$. However, for the approximate testing of the superconvergence relations at $u=0$ it is sufficient to assume that the interval $I_1 \cap I_2$ is near zero. This is plausible on the basis of an argument due to Sakmar(50), according to which if $Re\,\alpha(\sqrt{u})$ has minimum at $\sqrt{u} \neq 0$ then the trajectory $Re\,\alpha(u)$ not only coincides at $u=0$ with the corresponding MacDowell symmetric trajectory but also the two touch the line $u=0$, so that the situation on the Chew-Frautchi plot is roughly as shown in the figure.



It should be mentioned that the above conditions on u (eqns. 5.1,2,5) are sufficient and not necessary for the validity of eqns. (5.4,7). The domain of validity of the equations (5.4,7), may actually extend by analytic continuation outside the intervals $I_1 \cap I_2, I_1'$. We shall not touch upon the rather difficult question of the domain of validity of the superconvergence relations and shall be content with the assumption, made plausible in the foregoing discussion, that such a domain exists not very far from $u=0$.

We shall now use the approximation of retaining only the contributions of $N_{33}^*(1236)$, $N_{31}^*(1525)$ in the s channel and ρ, ω in the t channel to estimate the l.h.s. of equations (5.4,7), π, $A1$ in the t-channel make no contribution to any of the superconvergence relations given above.

The various $N^*_{33}N\gamma$ and $N^*_{31}N\gamma$ couplings are known from the analysis of experimental data on pion photoproduction(27). They are

$$\mathcal{L}_{int.}(N^*_{33}N\gamma) = \frac{iC_2}{m_\pi}\left[\bar{\psi}\gamma_\nu\gamma_5\psi_\mu + \bar{\psi}_\mu\gamma_\nu\gamma_5\psi\right]F^{\mu\nu}$$
$$+\frac{iC_4}{m_\pi^2}\left[\bar{\psi}\gamma_5\overleftrightarrow{\partial}_\nu\psi_\mu - \bar{\psi}_\mu\gamma_5\overleftrightarrow{\partial}_\nu\psi\right]F^{\mu\nu}$$

$$\mathcal{L}_{int.}(N^*_{31}N\gamma) = \frac{iD_2}{m_\pi}\left[\bar{\psi}\gamma_\nu\psi_\mu + \bar{\psi}_\mu\gamma_\nu\psi\right]F^{\mu\nu}$$
$$+\frac{iD_4}{m_\pi^2}\left[\bar{\psi}\overleftrightarrow{\partial}_\nu\psi_\mu + \bar{\psi}_\mu\overleftrightarrow{\partial}_\nu\psi\right]F^{\mu\nu}. \tag{5.8}$$

where $C_2 = .31,\ C_4 = -.0043,\ D_2 = .033,\ D_4 = .0117.$.

The coupling constants(*) for $N^*_{33}N\pi$ and $N^*_{31}N\pi$ interactions are determined from the widths of N^*_{33} , N^*_{31}

$$\langle N^+ p' |\, J^0_\pi(0)\, | N^{*+}_{33}\, p_n\rangle = \frac{i\lambda}{m_\pi}\, p'_\mu\, \bar{u}(p')\, u^\mu(p_n) \quad : \ \lambda = \sqrt{\tfrac{2}{3}}\times 2.0$$

$$\langle N^+ p' |\, J^0_\pi(0)\, | N^{*+}_{31}\, p_n\rangle = \frac{i\lambda'}{m_\pi}\, p'_\mu\, \bar{u}(p')\,\gamma_5\, u^\mu(p_n) \quad : \ \lambda' = 1.97$$

To calculate the $\rho N\bar{N}$ and $\omega N\bar{N}$ couplings we shall assume the ρ and ω dominance respectively of the isovector and isoscalar electromagnetic couplings of the nucleon, so that

$$\frac{1}{m_\omega^2}\, g_{\gamma\omega}\, F_{1,2}^{(\omega N\bar{N})} = F_{1,2}^{S} \ , \quad \frac{1}{m_\rho^2}\, g_{\gamma\rho}\, F_{1,2}^{\rho N\bar{N}} = F_{1,2}^{V} \tag{5.9}$$

The $\omega\pi\gamma$ and $\rho\pi\gamma$ couplings are estimated as follows:

(*) The constants are thus determined up to a factor .
The signs chosen here are those adopted in reference ( )
which ensure approximate validity for the sum rules
derived there.

$$g_{\omega\pi\gamma} = g_{\omega\rho\pi} \; g_{\rho\gamma} \frac{1}{m_\rho^2} .$$

$$g_{\rho\pi\gamma} = g_{\omega\rho\pi} \; g_{\omega\gamma} \frac{1}{m_\omega^2} . \tag{5.10}$$

where $\mathcal{L}_{int}(\omega\rho\pi) = i g_{\omega\rho\pi} \, \epsilon_{\mu\nu\lambda\sigma} \, \partial^\lambda \omega^\mu \, \underline{\rho}^\nu . \, \partial^\sigma \underline{\pi} .$

From the Gell-Mann, Sharp and Wagner model(42) for $\omega \rightarrow 3\pi$ decay we have

$$g_{\omega\rho\pi} = 51.0 \ \text{Bev}^{-1}$$

The ratio $\frac{g_{\gamma\omega}}{g_{\gamma\rho}}$ encountered in the preceding is estimated, under the assumption of F type of $VB\bar{B}$ interaction, to be

$$\frac{g_{\gamma\omega}}{g_{\gamma\rho}} = \frac{1}{\sqrt{3}} \; \frac{m_\omega^2}{m_\rho^2} \tag{5.11}$$

The F type of $VB\bar{B}$ interaction is indeed a natural generalisation of the vector theory of strong interactions, in which the ρ is coupled to the isovector current and the ω is coupled to the hyper charge current.

In the framework of the above described approximations, we get

$$m^2 \int Im \, F_{3,4}^{(3/2)} (\nu, 0) \, d\nu = -\frac{1}{3} g_{\pi N} F_1^V + \frac{1}{12} g_{\pi N} F_2^V - \frac{mM}{12 m_\pi^2} \lambda C \tag{5.12}$$
$$+ \frac{mM(m-M)}{12 m_\pi^3} \lambda D + \frac{mM_1(m+M_1)}{6 m_\pi^3} \lambda' D' - \frac{\sqrt{3}}{24} m \, g_{\omega\rho\pi} (F_2^S + 2 F_1^S)$$

$$m^2 \int Im \, F_{3,4}^{(\frac{1}{2}) \, \text{isovector}} (\nu, 0) \, d\nu = \frac{1}{12} g_{\pi N} F_1^V - \frac{1}{48} g_{\pi N} F_2^V - \frac{\lambda C}{6 m_\pi^2} mM \tag{5.13}$$
$$+ \frac{mM(m-M)}{6 m_\pi^3} . \lambda D - \frac{mM_1(m+M_1)}{24 m^3_\pi} \lambda' D' - \frac{\sqrt{3}}{48} m \, g_{\omega\rho\pi} (F_2^S + 2 F_1^S)$$

$$\sqrt{3}\, m^2 \int \mathrm{Im}\, F_{3,4}^{(\frac{1}{2})\ \text{isoscalar}}(\nu,0)\, d\nu = \frac{3}{4} g_{\pi N} F_1^S - \frac{3}{16} g_{\pi N} F_2^S - \frac{3}{8} \frac{m M_1 (m+M_1)}{m_\pi^3} \lambda' D' - \frac{\sqrt{3}}{8} m g_{\omega\rho\pi} \left(F_2^V + 2F_1^V\right) \tag{5.14}$$

where the contributions to $F_{3,4}^{(1/2)}(\nu,0)$ of the isoscalar and isovector components of the electromagnetic interactions have been estimated separately.

On comparison against numerical values we find

$$m^2 \int \mathrm{Im}\, F_{3,4}^{(3/2)}(\nu,0)\, d\nu = -4.50 + 4.16 - 2.12 + 0.06 + 5.26 - 2.22 = 0.64. \tag{5.15}$$

so that in the approximations under discussion we obtain verification for

$$\int \mathrm{Im}\, F_{3,4}^{(3/2)}(\nu,0)\, d\nu = 0$$

On the other hand, a similar statement cannot be made in the case of $\int \mathrm{Im}\, F_{3,4}^{(\frac{1}{2})}(\nu,0)\, d\nu = 0$. This indicates that either the relation is incorrect because of the possible erroneousness of the assumption (5.5) from which it is derived, (our present knowledge of the dominant fermion Regge trajectories may be far from complete), or less plausibly perhaps, the approximations used may be simply inadequate.

APPENDIX 1

Table of Hyperons and Hyperon Resonances

	I	J^P	Mass(MeV)	Mean life(sec)	Width	Decay mode	Fraction
$\Lambda(\Lambda_\alpha)$	0	$\frac{1}{2}^+$	1115.44 ± 0.12	$2.61 \pm .02 \times 10^{-10}$		$p\pi^-$	$66.3 \pm 1.0\%$
						$n\pi^0$	$33.6 \pm 1.0\%$
Σ^+	1	$\frac{1}{2}^+$	1189.39 ± 0.14	$0.794 \pm .026 \times 10^{-10}$		$p\pi^0$	$51.0 \pm 2.4\%$
						$n\pi^+$	$49.0 \pm 2.4\%$
Σ^0			1192.3 ± 0.2	$< 1.0 \times 10^{-14}$		$\Lambda\gamma$	100%
Σ^-			1197.20 ± 0.14	$1.58 \pm .05 \times 10^{-10}$		$n\pi^-$	100%
Ξ^0	$\frac{1}{2}$	$\frac{1}{2}^+$	1314.3 ± 1.0	$3.05 \pm .38 \times 10^{-10}$		$\Lambda\pi^0$	100%
Ξ^-			1320.8 ± 0.2	$1.75 \pm .05 \times 10^{-10}$		$\Lambda\pi^-$	100%
						$\Lambda e^-\nu$	$< 1.7 \times 10^{-}$
$Y_0^*(\Lambda_\beta)$	0	$\frac{1}{2}^-$	1405	-	35 ± 5	$\Sigma\pi$	100%
						$\Lambda\pi\pi$	$< 1\%$.
$Y_0^*(\Lambda_\gamma)$	0	$\frac{3}{2}^-$	1518.9 ± 1.5		16 ± 2	$\Sigma\pi$	$\sim 55 \pm 7$
						$\bar{K}N$	$\sim 29 \pm 4$
						$\Lambda\pi\pi$	$\sim 15 \pm 2$
$Y_0^*(\Lambda_\alpha^2)$	0	$\frac{5}{2}^+$	1815 ± 5		50	$\bar{K}N$	~ 75
						$\Sigma\pi$	~ 9
						$\Lambda\pi\pi$	~ 15
$Y_1^*(\Sigma_\delta)$	1	$\frac{3}{2}^+$	$1382.7 \pm .5$		44 ± 2	$\Lambda\pi$	$\sim 90 \pm 2$
						$\Sigma\pi$	$\sim 10 \pm 2$
$Y_1^*(\Sigma_\beta')$	1	$\frac{3}{2}$	1660 ± 10		44 ± 5	$\bar{K}N$	~ 15
						$\Sigma\pi$	~ 30
						$\Lambda\pi$	~ 5
						$\Sigma\pi\pi$	~ 30
						$\Lambda\pi\pi$	~ 20

$Y_1^*(\Sigma_\delta^{II})$	1	$\frac{7}{2}^+$	2065		~ 160	$\bar{K}N$ ~ 35% $\Lambda\pi$
$\Xi^*(\Xi_\delta)$	$\frac{1}{2}$	$\frac{3}{2}^+$	1529.7 ± 0.9		7.5 ± 1.7	$\Xi\pi$ ~ 100%
$\Xi^*(\Xi_\gamma)$	$\frac{1}{2}$	$\frac{3}{2}^-$	1816 ± 3		~ 16 ± 4	$\Xi^*\pi$ ~ 25 $\bar{K}\Lambda$ ~ 65 $\Xi\pi$ ~ 5 $\Xi\pi\pi$ ~ 5
$\Xi^*(\Xi_\alpha^2)$	$\frac{1}{2}$	$\frac{5}{2}^+$	1933 ± 16		140 ± 35	$\Xi\pi$
Ω^-	0	$\frac{3}{2}^+$	1675 ± 3	1.3×10^{-10}		$\Xi\pi$. $\bar{K}\Lambda$.

APPENDIX 2

Definition of Form Factors:

All the form factors appearing in the following are real on account of the requirement of symmetry under time inversion(7).

The notation used in the following for the hyperon resonances is the same as the one adapted in the tables of Rosenfeld et. al. and is explained in Appendix 1.

The matrix elements for the pion source J_π are defined as

$$\langle \Sigma p' i | J_\pi^k(0) | \Sigma p_n l \rangle = K_{\Sigma\Sigma\pi}(q^2)\, \bar{u}(p')\, i\gamma_5\, u(p_n)\; i\epsilon_{kli} \qquad q = p_n - p'$$

$$\langle \Sigma p' i | J_\pi^k(0) | \Lambda p_n \rangle = K_{\Sigma\Lambda\pi}(q^2)\, \bar{u}(p')\, i\gamma_5\, u(p_n)\, \delta_{ik}$$

$$\langle \Sigma p' i | J_\pi^k(0) | \Lambda_\beta p_n \rangle = i K_{\Sigma\Lambda_\beta\pi}(q^2)\, \bar{u}(p')\, u(p_n)\, \delta_{ik}$$

$$\langle \Sigma p' i | J_\pi^k(0) | \Sigma_\delta p_n l \rangle = K_{\Sigma\Sigma_\delta\pi}(q^2)\, \bar{u}(p')\, p'_\mu\, u^\mu(p_n)\, i\epsilon_{kli}$$

$$\langle \Sigma p' i | J_\pi^k(0) | \Lambda_\gamma p_n \rangle = K_{\Sigma\Lambda_\gamma\pi}(q^2)\, \bar{u}(p')\gamma_5\, p'_\mu\, u^\mu(p_n)\, \delta_{ik}$$

$$\langle \Sigma p' i | J_\pi^k(0) | \Lambda_\alpha^2 p_n \rangle = i K_{\Sigma\Lambda_\alpha^2\pi}(q^2)\, \bar{u}(p')\, p'_\lambda\, p'_\mu\, \gamma_5\, u^{\lambda\mu}(p_n) \cdot \delta_{ik}$$

$$\langle \Sigma p' i | J_\pi^k(0) | \Sigma_\rho p_n l \rangle = i K_{\Sigma\Sigma_\rho\pi}(q^2)\, \bar{u}(p')\, p'_\lambda\, p'_\mu\, u^{\lambda\mu}(p_n)\, i\epsilon_{kli}$$

$$\langle \Lambda p' | J_\pi^k(0) | \Sigma_\delta p_n l \rangle = K_{\Lambda\Sigma_\delta\pi}(q^2)\, \bar{u}(p')\, p'_\mu\, u^\mu(p_n)\, \delta_{kl}.$$

$$\langle \Lambda p' | J_\pi^k(0) | \Sigma'_\rho p_n l \rangle = K_{\Sigma'_\rho\Lambda\pi}(q^2)\, \bar{u}(p')\, p'_\mu\, \gamma_5\, u^\mu(p_n)\, \delta_{kl}.$$

$$\langle \Lambda p' | J_\pi^k(0) | \Sigma_\rho, p_n, l \rangle = i K_{\Lambda\Sigma_\rho\pi}(q^2)\, \bar{u}(p')\, p'_\lambda\, p'_\mu\, u^{\lambda\mu}(p_n)\, \delta_{kl}$$

$$\langle \Xi p' | J^k_\pi(0) | \Xi p_n \rangle = K_{\Xi\Xi\pi}(q^2)\, \bar{u}(p')\, i\gamma_5\, u(p_n)\; \chi^T \tau^k \chi$$

$$\langle \Xi p' | J^k_\pi(0) | \Xi_\delta p_n \rangle = K_{\Xi_\delta\Xi\pi}(q^2)\, \bar{u}(p')\, p'_\mu u^\mu(p_n)\; \chi^T \tau^k \chi .$$

$$\langle \Xi p' | J^k_\pi(0) | \Xi_\gamma p_n \rangle = K_{\Xi\Xi_\gamma\pi}(q^2)\, \bar{u}(p')\, \gamma_5\, p'_\mu u^\mu(p_n)\; \chi^T \tau^k \chi .$$

$$\langle \Xi p' | J^k_\pi(0) | \Xi_\alpha p_n \rangle = i K_{\Xi\Xi_\alpha\pi}(q^2)\, \bar{u}(p')\, p'_\lambda p'_\mu\, \gamma_5\, u^{\lambda\mu}(p_n) . \chi^T \tau^k \chi .$$

$u(p)$, $u_\mu(p)$, $u_{\mu\nu}(p)$ are the spinor wave functions for particles of spins $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ respectively, so that

$(\not{p} - im)\, u(p) = 0$ etc., where m is the mass of the particle.

$\gamma^\mu u_\mu(p) = 0$ $\qquad \gamma^\mu u_{\mu\nu}(p) = 0$

χ is a two component spinor in isospin space.

The matrix elements for the electromagnetic current V_μ are similarly defined as

$$\langle \Sigma p_n i | V_\mu(0) | \Sigma p j \rangle = i\, \bar{u}(p_n) \left[F_1(k^2)\gamma_\mu - \frac{\sigma_{\mu\nu} k^\nu}{2m} F_2(k^2) \right] u(p)$$

$k = p_n - p$, $\sigma_{\mu\nu} = -\frac{i}{2}[\gamma_\mu$

$$F_{1,2}(k^2) = F^S_{1,2}(k^2)\,\delta_{ij} + i\, F^V_{1,2}(k^2)\,\epsilon_{3ji} .$$

$$\langle \Lambda p_n | V_\mu(0) | \Lambda p \rangle = i\, \bar{u}(p_n) \left[F_1^\Lambda(k^2)\gamma_\mu - \frac{\sigma_{\mu\nu} k^\nu}{2m} F_2^\Lambda(k^2) \right] u(p)$$

$$\langle \Lambda p_n | V_\mu(0) | \Sigma p j \rangle = -i\, \bar{u}(p_n)\, \frac{\sigma_{\mu\nu} k^\nu}{2m} F^{\Sigma\Lambda}(k^2)\, u(p)\, \delta_{3j} .$$

$$\langle \Lambda_\beta p_n | V_\mu(0) | \Lambda p \rangle = -i\, \bar{u}(p_n)\, \frac{\sigma_{\mu\nu} k^\nu}{2m} H_{\Lambda\Lambda_\beta}(k^2)\, \gamma_5\, u(p) \qquad \text{etc.}$$

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